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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.

2. To supply an additional medium for the publication of expository mathematical articles.

3. To promote more scientific methods of teaching mathematics.

4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Does Mathematics Work?

One evening Thorburn Reid addressed The American Institute of Electrical Engineers on the important subject of *The Armature Reactions of Alternators*, but, in treating the subject, purposely omitted any reference to "the third harmonics." He had not been able to make the mathematics work here in connection with his theorems on electricity. Charles P. Steinmetz, holding a doctorate in mathematics, but as yet in obscurity, occupied a seat on the back row. Upon the conclusion of Reid's address, he arose and, in the name of mathematics, entered vigorous protest. At the next regular meeting of the club, he showed that mathematics did work where Reid had not been able to use it. Reid and Steinmetz became great friends. Steinmetz had started upon a great career devoted to the proposition that, at least in electricity, "mathematics does work."

Recently a group of business men and bankers sat in annual convention. A prominent lecturer had the floor and in his address, more than once paused to say, "But here mathematics does not work." Yet, in all the instances cited mathematics *actually does* work.

A principle of the highest importance stands out in these cases! The mathematician has the machinery needed by the public, but the public does not always know of its existence. Obviously then the mathematician has something more to do than to discover laws of science and of business and to formulate them in terms of mathematics. It is not enough for him to stand before conventions and to proclaim to his mathematical colleagues the importance of mathematics. He must realize that a widened appreciation of the values and service-functions of this science is the consequence of its having become the property of those who need and can use it.

The mathematician must share in the promotion of programs directed to bringing about a greater public acquaintance with mathematics and a wider use of it. The automobile is an exceedingly useful machine—but *only* to him who can command its services: and the auto-dealer has discovered that free lessons in "How to drive a car" increases sales.

Mathematics will work, but only for him who knows how to command its services.

Irby C. Nichols.

Three-Circle Problems in Modern Geometry*

By C. D. Smith
Mississippi State College

1. *Introduction.* Important problems associated with the triangle have been discussed by writers of different periods and from different points of view. The result is a variety of approaches to apparently unrelated problems. The main purpose of this paper is to show by means of certain three-circle constructions that a number of these problems are very closely related. Certain important relations appear which seem not to have been mentioned elsewhere. Among the well known propositions which have been proved by three-circle constructions or which follow directly from such proofs are, A Theorem of Miquel, the Simon Line, and the Brocard Points.† We will demonstrate certain other well known theorems by three-circle methods, thereby showing that all are related parts of the same system.

2. *The Fundamental Theorem.* It is well known that the three chords, determined by three coplanar circles which meet two by two, have a point in common which is called the radical center of the system. Since the opposite angles of a convex cyclic quadrilateral are supplementary we begin with the following fundamental diagram.

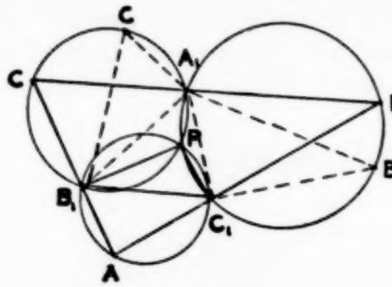


FIG. I

Construct circles O_1, O_2, O_3 as in Figure 1 and such that each angle at P is less than a straight angle. Inscribe the angles C_1AB_1 in circle O_1, A_1BC_1 in circle O_2, and B_1CA_1 in circle O_3. Each circle contains an

*Presented to the American Mathematical Society, July 5, 1938.
†Modern Geometry, by R. A. Johnson (1929).

inscribed convex quadrilateral with a common vertex at P. Angles A, B, C, have their respective supplements at P. Hence the sum of angles A, B, and C is equivalent to two right angles. The result is

The Fundamental Theorem 1. If three coplanar circles have a point in common and meet again two by two, so that the common chords form an inscribed angle less than a straight angle in each circle, and if an angle is inscribed in each of the conjugate arcs, the sum of these three inscribed angles is equivalent to two right angles.

As a special case, if points A, C_1, B are collinear and points B, A_1, C are collinear, then points C, B_1, A are also collinear and ABC is a triangle, because the sum of the angles of a triangle is equivalent to two right angles. Conversely, if circles AB_1C_1 and BC_1A_1 relative to triangle ABC meet at P, then circle CA_1B_1 must contain P because $\angle A_1PB_1$ is the supplement of $\angle C$. Hence we have a theorem of Miquel and its converse.

A Theorem of Miquel 2. If a point on each side of a triangle is fixed and a circle is drawn through each vertex and the two fixed points on the sides of that angle, the three circles thus determined have a point in common. And conversely, if three circles meet in a point interior to a triangle and each circle contains one and only one vertex of the triangle, the circles meet two by two on the respective sides of the triangle.

Many important consequences of the Miquel construction have been published.* It is well known that a Miquel point exists when one or more of the selected points fall at the vertices or on the prolongations of the sides of the triangle of reference. Also the Miquel point may fall exterior to the triangle. Certain important types of one to one correspondence exist between systems of interior and exterior Miquel points which will be made the basis of discussion in another paper.

3. *Discussion of the Theorem of Ceva.*

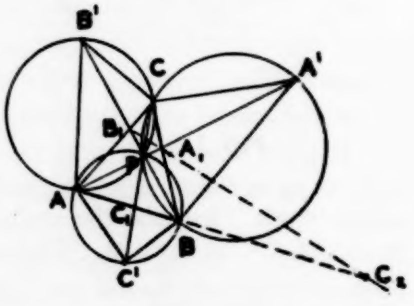


FIG. 2

*Johnson, *loc cit.*, 1, Ch. VII.

Draw circles APB, BPC, and CPA where P is any point interior to the triangle, as in Figure 2. Produce AP to meet BC at A_1 and to meet circle BPC at A'. Produce BP and CP in a similar manner and complete the diagram. In circle BPC we have, $\angle A'CB = \angle A'PB$, and $\angle A'BC = \angle A'PC$. Hence triangles $A'CA_1$ and BPA_1 are similar. Also triangles $A'BA_1$ and PCA_1 are similar. From analogous results in circles APC and APB we have by proportional parts.

$$AC_1/BC_1 = AC' \cdot PA/BC' \cdot PB;$$

$$BA_1/CA_1 = BA' \cdot PB/CA' \cdot PC;$$

$$CB_1/AB_1 = CB' \cdot PC/AB' \cdot PA.$$

The product of corresponding members of the equations gives

$$AC_1 \cdot BA_1 \cdot CB_1/BC_1 \cdot CA_1 \cdot AB_1 = AC' \cdot BA' \cdot CB'/BC' \cdot CA' \cdot AB'.$$

From similar triangles we have at point C,

$$\angle A'CB = (\angle A_1PB) = \angle B'CA,$$

and analogous relations hold relative to points A and B. Since the sides of a triangle are proportional to the sines of the respectively opposite angles it follows directly that $AC' \cdot BA' \cdot CB'/BC' \cdot CA' \cdot AB' = 1$. The result is

Ceva's Theorem 3. If point P is interior to triangle ABC and if AP meets BC at A_1, BP meets CA at B_1, and CP meets AB at C_1, then the numerical ratio $AC_1 \cdot BA_1 \cdot CB_1/BC_1 \cdot CA_1 \cdot AB_1$ is equal to one.

The analogous relation holds when P is exterior to the triangle. In particular when C_2 is the harmonic conjugate of C_1 the point P is on the circumcircle and B_1, A_1, and C_2 are collinear.* The result, of course, is the same as that given by the theorem of Menelaus whereby a straight line divides the sides of a triangle so that the product of the three ratios is one.

4. *The Nine Point Circle and Associated Theorems*

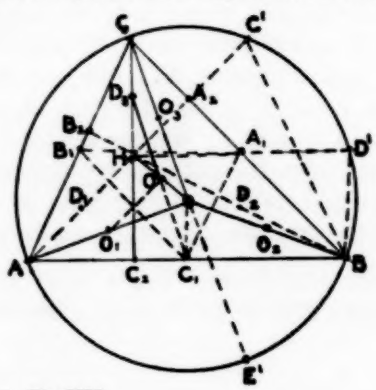


FIG. 3

*Johnson, *loc cit.*, 1. Ch. VII.

In Figure 3 with the circumcenter O of triangle ABC as Miquel point and with OA, OB, and OC as diameters, the three circles O_1, O_2, O_3, determine the midpoints, C_1 of AB, A_1 of BC, and B_1 of CA. Triangle $A_1B_1C_1$ divides the given triangle into four congruent triangles, the circle $A_1B_1C_1$ is congruent to each of the three given circles, and its radius is half the radius of the circumcircle ABC. Next with the orthocenter H as Miquel point, with AH, BH, and CH as diameters, and with D_1, D_2, D_3, as corresponding centers, the radical axes are the altitudes AA_2, BB_2, and CC_2. Join O to H and with O' as midpoint draw $O'D_3$. Line O_1O' joins midpoints of two sides of triangle AOH and is therefore the perpendicular bisector of B_1C_1. In like manner O_2O' is the perpendicular bisector of C_1A_1, and hence O' is the center of circle $A_1B_1C_1$. We have

Theorem 4. If the Miquel circle of the circumcenter is a pedal circle, its center is the midpoint of the Euler line segment from the circumcenter to the orthocenter of the triangle.

Moreover, $O'D_3$ is a radius of circle O' since $O'D_3 = OC/2$. Similar results for D_1 and D_2 place the centers of the three circles which have H for Miquel point on circle O'. Now $D_3O'C_1$ is a diameter of circle O' because OC_1 is parallel to HD_3. But $C_1C_2D_3$ is a right triangle with the right angle at C_2, hence C_2 falls on circle O'. Similar results for A_2 and B_2 give

The Nine-point Circle Theorem 5. If the Miquel circle of the circumcenter is a pedal circle, it contains the midpoints of the sides, the feet of the altitudes, and the centers of three circles whose diameters are the distances from the orthocenter to the vertices.

Next with H as center of similitude and $K = 2$ as ratio of similitude let point C move along the circumcircle. Point D_3 moves along circle O' and when C comes to C' point D_3 comes to A_2. Now triangles CA_2C' and AA_2B are similar and triangles HCC' and HBC' are isosceles. Prolong HA_1 to meet the circumcircle at D' and draw $D'B$. Triangles A_1BD' and A_1CH are congruent, since A_1 is the midpoint of both BC and HD'. Hence $D'B$ is perpendicular to AB and $D'OA$ is a diameter of the circle ABC. In like manner HC_1 will pass through the end of the diameter CE' and $ACD'E'$ is an inscribed rectangle. Analogous results hold for sides AB and BC. We can now state

Theorem 6. Lines from the orthocenter through the mid-points of two sides of a triangle meet the circumcircle in two points which, with the third side of the triangle, determine an inscribed rectangle.

5. *Isogonal Points and Lines*. Begin with triangle ABC and circles APB, BPC, and CPA, where P is an interior Miquel point, as in Figure 4.

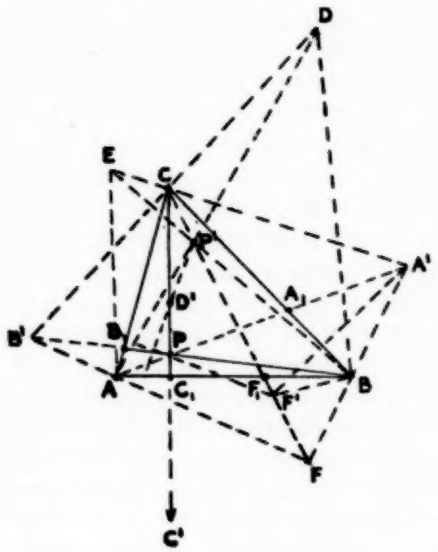


FIG. 4

Prolong AP to meet BC at A_1 and to meet circle BPC at A'.

Prolong BP to meet CA at B_1 and to meet circle CPA at B'.

Prolong CP to meet AB at C_1 and to meet circle APB at C'.

Make a similar construction for P', the isogonal conjugate of P.* Complete the diagram so that CP' produced meets the circle CPB at F' and CP meets the circle $AP'C$ at D'. Draw $P'D'$ and PF'. Obviously the relations established in Figure 2 apply to either system of three circles in Figure 4. Moreover, $\angle BPA' = \angle BCA'$, $\angle BPC' = \angle PCB + \angle PBC$, and $\angle C'PA = \angle PCA + \angle PAC$. By addition ECA' is a straight angle, since

$$\angle PBC + \angle PAC = \angle BAP' + \angle ABP' = \angle AP'E = \angle ACE.$$

In a similar manner $B'CD$ is a straight line. Likewise the corresponding lines at vertices A and B are straight lines and an interesting variety of pairs of similar triangles appear. For example, triangles APB and ECB are similar; triangles BPC and CAF are similar; and triangles CPA and ABD are similar. The corresponding pairs of triangles with reference to P' are similar.

If A' moves on circle CPB in clockwise fashion until it falls on vertex B of triangle ABC, the line $A'B$ becomes tangent to the circle

*Johnson, *loc. cit.*, 1, p. 154.

at B and forms with BC an angle equal to angle A' of the quadrilateral $CPBA'$. Similar results for the vertices A and C establish the fact that tangents to the circles at the vertices of triangle ABC, when taken in clockwise fashion form angles with the respective sides such that the sum is equivalent to two right angles. If the movements are made counter clockwise we obtain a similar result. We now make use of this method and the same lettering as in Figure 4 to represent a special construction which leads to the Brocard points and other points associated with them. Let a perpendicular to AC at C meet the perpendicular bisector of BC and with the intersection as center and BC as chord draw a circle. Side AC is now tangent to this circle at C. Likewise draw a circle on AC as chord and tangent to AB at A. Let the two circles meet at P and a third circle APB is then tangent to BC at B. In this special case the angles A', B', C', of the given inscribed quadrilaterals are equal respectively to angles C, A, B, of the triangle. Also $\angle PCA = \angle PAB = \angle PBC$. When points A, P, A_1, and A' are collinear as in the diagram we have

$$\angle PA'C = \angle PBC = \angle PCA = \angle PAB.$$

Hence $A'C$ is parallel to AB and $\angle ABC = \angle A'CB = \angle A'PB$. Triangles AA_1B and BPA_1 are similar, and triangles BCA' and ABC are similar. Analogous relations hold with reference to circles CPA and APB.

Next let us reverse the order in which the circles are drawn and determine the center of the first circle by letting a perpendicular to BC at C meet the perpendicular bisector of AC. Then the circle with AC as chord is tangent to BC at C. Likewise draw a circle on BC as chord tangent to AB at B. Let the two circles meet at a point P' and it follows that $\angle P'BA = \angle P'CB$. The third circle $AP'B$ is tangent to AC at A since $\angle P'BA = \angle P'AC$. The properties of this system are analogous to those mentioned for the three circles of the clockwise system. Moreover P and P' are isogonal conjugates for if Q is the isogonal conjugate of P we have $\angle QCB = \angle PCA$ and likewise at vertices A and B. It follows at once that $\angle QCB = \angle QAC = \angle QBA$ and Q coincides with P' since the construction is unique. These two points P and P' have been called the Brocard points of the triangle, although they were mentioned as early as 1816 by Crelle, Jacobi, and others.*

In addition to the above, we find the following interesting properties in the figure. The triangles BCA' and ACE are each similar to ABC. Prolong CP' to circle CPB at F', let CP meet circle $CP'A$ at

*Johnson, *loc cit.*, 1, p. 263.

D', and draw $F'B$, $F'A'$, $F'P$, and $D'P'$. Also triangle $AP'C$ is similar to $CF'B$, and triangle $CP'D'$ is similar to both $PF'B$ and $F'PC$. Triangle ECP' is similar to $F'BA'$, and triangle APB is similar to $CP'B$. Line PF' is parallel to BC, and $P'D'$ is parallel to AC. The two lines form an angle equal to angle C of the triangle ABC. The points D' and F' are associated with the Brocard points P and P' by

Theorem 7. Let the Brocard rays from one angle of a triangle be prolonged to meet the circles which contain the given vertex and the corresponding Brocard point and join the extremities of these chords to the Brocard points. Then the chords thus formed are each parallel to one side of the given triangle and form similar triangles with the two Brocard rays.

If we begin with each of the vertices A and B and locate the two corresponding points given in the theorem, we have in all six points associated by pairs with the Brocard points. Such pairs of points may be called Brocard conjugates of P and P'. We have located the Brocard points as special cases of isogonal conjugates which are in turn special cases of Miquel points. Moreover the method of three circles gives a variety of additional relations including three pairs of points associated with the Brocard points of the triangle.

6. *The Nagel Point and Associated Points.*

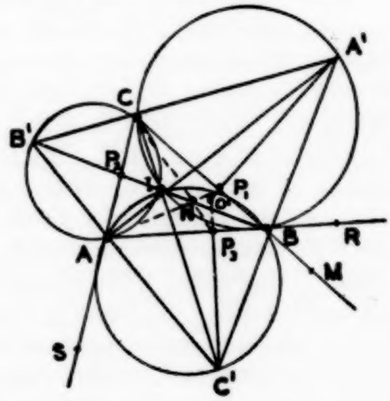


FIG. 5

For a given triangle ABC with incenter I, draw the three circles, as in Figure 5. Prolong AI to A' on circle BIC and draw $A'B$ and $A'C$. In like manner let CI determine C' on circle AIB and let BI determine B' on circle CIA. Then $\angle BIA' = \angle BCA'$, and also

$$\angle BIA' = \angle IAB + \angle IBA.$$

Hence $\angle ICA'$ is a right angle because it is equivalent to half the sum of the angles of the triangle. Hence IA' is a diameter of circle BIC. Similar relations hold for circles AIB and CIA and the respective diameters are IA', IB', IC'. This means that $A'B'C'$ is a triangle with sides respectively perpendicular to the angle bisectors of triangle ABC. The Miquel point I is orthocenter of $A'B'C'$ and ABC is the Miquel pedal triangle. Let a circle on $C'B$ as diameter meet AB at P_3 and CB produced at M. Let a circle on $A'B$ as diameter meet BC at P_1 and AB produced at R. Also let a circle on $B'C$ as diameter meet AC at P_2 and a circle on $C'A$ as diameter meet CA produced at S. The right triangles $CC'M$ and $CC'S$ are congruent, $C'BP_3$ and $C'BM$ are congruent, and $C'AS$ and $C'AP_3$ are congruent. As a result we have $CA+AP_3=CB+BP_3$. In like manner the right triangles $A'BP_1$ and $A'BR$ are congruent and a similar argument gives $AB+BP_1=AC+CP_1$, and also $AB+AP_2=BC+CP_2$. Hence

$$AP_3+P_3B+BP_1=CP_1+BP_1+P_3B, \quad \text{or} \quad AP_3=CP_1.$$

Likewise $BP_1=AP_2$, and $BP_3=CP_2$. The product of the last three equalities gives $AP_3 \cdot BP_1 \cdot CP_2=P_3B \cdot CP_1 \cdot AP_2$ and we know from the theorem of Ceva that lines from P_1, P_2, P_3, to the respectively opposite vertices of triangle ABC meet in a point N, which we recognize as the Nagel point.*

Note that we have associated point N directly with I as Miquel point. Circles on IA, IB, IC, as diameters determine the pedal triangle $I_1I_2I_3$. Circles with diameters II_1, II_2, II_3, determine triangle $J_1J_2J_3$ with I as orthocenter. The altitudes J_1A_1, J_2B_1, J_3C_1, are on the original angle bisectors of triangle ABC. We now have a system in which triangle $A_1B_1C_1$ corresponds to ABC and triangle $J_1J_2J_3$ corresponds to $A'B'C'$. Hence triangle $J_1J_2J_3$ determines a Nagel point N' of triangle $A_1B_1C_1$ which is also associated with the original Miquel point I of ABC. Beginning with triangle $J_1J_2J_3$ we may repeat the cycle which began with $A'B'C'$ and by continuing the process we produce a set of Nagel points associated with I as Miquel point. In the system thus generated, those triangles analogous to $J_1J_2J_3$ have I for orthocenter, those analogous to $I_1I_2I_3$ have I for circumcenter, and those analogous to $A_1B_1C_1$ have I for incenter. We may begin by drawing parallels to the respective sides of triangle $A'B'C'$ through the vertices and develop the system where the triangles of each cycle are enlarged. Note that we have an infinite set of Nagel points associated with I by triangles which recur in cycles such that I is respectively orthocenter, incenter, and circumcenter.

*A Treatise on the Circle and the Sphere, by J. L. Coolidge.

Another interesting set of points associated with I may be illustrated by the circumcenter of triangle $A'B'C'$. For if $A'P_1$ and $C'P_3$ meet at O' the three circles which have $P_1P_2P_3$ for Miquel triangle have $O'A$, $O'B$, $O'C$, for diameters. $\angle P_1BA' = \angle P_3BC'$ and hence $\angle P_1A'B = \angle P_3C'B$ which makes $O'A'C'$ an isosceles triangle. Likewise we may prove that triangle $O'A'B'$ is isosceles and it follows that O' is the circumcenter of $A'B'C'$. It is well known that the circumcenter O of triangle ABC bisects the segment IO'. Hence our system of recurring triangles will produce other points analogous to O' which reproduce this property.

We have shown in this paper how the method of three circles may be used to correlate the geometry of a large group of well known propositions and how the constructions may serve as a basis for a variety of additional relations.

Most people have heard only of "the calculus," a slip-shod expression applied to the infinitesimal calculus of mathematics invented by Leibniz and Newton. That calculus is undoubtedly the most impressive and important ever constructed, and we may forgive its admirers for trying to pre-empt the word "calculus" as its proper name; yet this word is too useful to be lost from the more general science of logic. A calculus is, in fact, any system wherein we may calculate. Ordinary arithmetic, the system of natural numbers with its constituent operations $+$, $\times$, $\div$, and $-$, is a calculus; the famous "hedonistic calculus" of Jeremy Bentham was so named in the fond and sanguine belief that this philosophy furnished a system wherein the relative magnitudes of pleasures could be exactly calculated. But as it involved no operations upon the elements called "pleasures," it failed to be a calculus.—From Susanne K. Langer's *Introduction to Symbolic Logic*. Published by Houghton Mifflin Company, 1937.

A Note to the Theory of Equations

By H. T. R. Aude
Colgate University

In the January, 1936 number of the National Mathematics Magazine, Geo. A. Yanosik gives an interesting and effective method for obtaining graphically the complex roots of a cubic. He closes his paper, however, with the statements: that the method applies only to a cubic with one real root; and that it does not apply to higher degree equations. Of these two statements the latter is true as the theorem stands, while the former is unnecessarily limited. It therefore seems that both merit further consideration, and it is here that this paper begins. First, by a re-examination of the facts it is shown that the method, there exhibited, does apply to any cubic. Thereupon the statements, or their implications, are viewed from a different point which lead to the converse theorem. Finally, this latter shows the way to a general theorem which does hold for higher degree equations.

To find graphically the roots of a cubic equation $F(x)=0$ it is sufficient, if all the roots are real, to view the graph $y=F(x)$ and read from this the x-intercepts. If, however, two roots are complex, then Yanosik's method solves the problem. It is therefore only a matter of interest, and we have the satisfaction that comes from a generalization, when we show that this last method holds for all cubics.

Consider a cubic equation and denote the three roots by r_1, r_2, and r_3. No two of these are assumed equal. For the cases where two or three roots are equal are simple, if not trivial. It is suitable and implies no loss of generality, to represent the cubic function by the equation

$$(1) \qquad y=F(x)=(x-r_1)(x-r_2)(x-r_3).$$

Since at least one of the zeros, say $x=r_1$, is real, a tangent t is drawn to one of the bulges of the curve from this point. This tangent is different from the one drawn to the curve at the point r_1, 0). Except in the case when $r_1=(r_2+r_3)/2$. The equation of t is seen to be

$$(2) \qquad y=F'(x_1)(x-r_1),$$

where P_1 the point of tangency has the coordinates (x_1, y_1). Eliminating the expression $y/(x-r_1)$ between equations (1) and (2) brings forth the quadratic equation

$$(3) \qquad x^2-(r_2+r_3)x+r_2r_3-F'(x_1)=0$$

of which $x=x_1$ is a double root. The theorems for the sum and the product of the roots give, respectively

$$(4)\qquad x_1=(r_2+r_3)/2 \quad\text{and}\quad x_1^2=r_2r_3-F'(x_1).$$

From these two equations, symmetric in r_2 and r_3, it is seen that

$$(5)\qquad (r_2-r_3)/2=\pm[-F'(x_1)]^{\frac{1}{2}}.$$

If it be agreed that $r_2>r_3$ when both roots are real, or that r_2 be the root with the positive coefficient of i when both roots are complex, then only the plus sign is used in (5) and this equation with the first equation in (4) yield

$$(6)\qquad r_2=x_1+[-F'(x_1)]^{\frac{1}{2}},\quad r_3=x_1-[-F'(x_1)]^{\frac{1}{2}}.$$

This is a symbolic statement of Yanosik's theorem, applicable however to any cubic. For, if the roots are all real and different, then the bulges of the cubic curve (1) are such that $F'(x_1)$, the slope of the tangent drawn from the point $(r_1, 0)$ to the curve, is a negative number. Therefore, the theorem

If a cubic function has the three zeros r_1, r_2, r_3 of which at least one, say r_1, is real, then the line drawn on the graph from the point $(r_1, 0)$ tangent to a bulge of the cubic curve is such that x_1, the abscissa of the point of tangency; is $x_1=(r_2+r_3)/2$; while $F'(x_1)$, the slope of the tangent, satisfies the relation $(r_2-r_3)/2=[-F'(x_1)]^{\frac{1}{2}}$.

Looking back upon the relations established it is noted that if the cubic $F(x)$ be suppressed by the division of $x-r_1$ to the quadratic $f(x)=(x-r_2)(x-r_3)$, then x_1, the abscissa of the point of tangency, is the one and only value of x for which $f'(x)$ the derivative of $f(x)$ vanishes. Also, it is seen from equation (3) that $F'(x_1)$ the slope of the tangent is such that

$$F'(x_1)=(x_1-r_2)(x_1-r_3)=f(x_1).$$

The value of this slope is the same number as the ordinate $f(x_1)$, and it is independent of r_1. Therefore, if one considered to the same axes the graphs of several cubic functions of the form

$$y=F(x)=(x-r_1)f(x),\quad r_1=\text{(say)}\ -3,\ 0,\ 2,\ 7$$

while the zeros of the quadratic function $f(x)=(x-r_2)(x-r_3)$ remain fixed, then it is true that the tangents to these cubics at the point on each where $f'(x)=0$ are parallel lines. Their slope is $f(x_1)$ while their x-intercepts are, respectively, the values selected for r_1.

These facts point the way to a theorem which holds for the graphs of those functions of higher degree which have at least one real zero. Write as before

$$(7)\qquad y=F(x)=(x-r_1)f(x)$$

but let $F(x)$ represent the function

$$F(x) = x^n + a_1 x^{n-1} + \cdots + a_n$$

of which one real zero is $x = r_1$. Then, $f(x)$ will be a function of degree $n-1$. Its derivative $f'(x)$ will have k real zeros $x = x_i$, $i = 1, \cdots, k$ where $k \lesseqgtr n-2$. The equation of the tangent to the curve (7) at any point $P_i(x_i, y_i)$ is

$$(8) \qquad y - y_i = F'(x_i)(x - x_i)$$

Differentiating (7) gives

$$F'(x) = f(x) + (x - r_1)f'(x).$$

Imposing the condition that $f'(x_i) = 0$ brings forth the relation

$$(9) \qquad F'(x_i) = f(x_i).$$

Turning to (7) with the coordinates of the point P_i gives

$$y_i = (x_i - r) \cdot f(x_i).$$

This value of y_i together with (9) applied to the tangent equation in (8) gives the result

$$(10) \qquad y = F'(x_i)(x - r_1) = f(x_i)(x - r_1).$$

Equation (10) holds for every one of the k tangents. It is thus seen that the k tangents drawn to the curve $y = F(x)$ at the k points where $f'(x) = 0$ are such that they all meet at the point $(r_1, 0)$. Whence the theorem

Any function $y = F(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ of degree three, or higher, which has at least one real zero at $x = r_1$ is such that the k real zeros of

$$\frac{d}{dx}\,[F(x)/(x - r_1)] \quad x = x_i \ (i = 1, \cdots, k)$$

mark points on the graph such that the k tangents drawn to the graph at the points $x = x_i$ are concurrent in the point $(r_1, 0)$.

A Note on the Diophantine Equation $Ax^2+y^2=z^2$

By WILLIAM H. ERSKINE
Bethany College

The following derivation of all primitive solutions of the diophantine equation $Ax^2+y^2=z^2$, following a geometrical method suggested by Klein* may be of interest to some of the readers.

Problem: To find all primitive integral solutions x,y,z of $Ax^2+y^2=z^2$, where A is a positive integer not factorable by the square of an integer. A solution x,y,z is called primitive if x, y, and z have no common factor.

The problem is related to that of finding all rational solutions ξ, η of $A\xi^2+\eta^2=1$ where

$$\xi=\frac{x}{z}, \quad \eta=\frac{y}{z}$$

and it is in this form that we approach it. We introduce a rectangular coordinate system and call a point (ξ, η) rational if both of its coordinates ξ and η are rational. We interpret the problem as that of finding all rational points on the ellipse $A\xi^2+\eta^2=1$.

We note at sight that $\xi=0$, $\eta=-1$ is a rational point of the ellipse. and that a line determined by this point and any other rational point will have a rational slope. We consider, therefore, rays through $(0,-1)$ with rational slope λ and call these rational rays. The equation of such a ray will be $\eta+1=\lambda\xi$.

The rational rays through $(0,-1)$ will intersect the ellipse in the point

$$\left(\frac{2\lambda}{\lambda^2+A}, \quad \frac{\lambda^2-A}{\lambda^2+A}\right)$$

and we note that for each rational λ this will be a rational point. Moreover, every rational point of the ellipse will have coordinates of this form.

*Klein: *Elementary Mathematics from an Advanced Standpoint.* Translation by Hedrick and Noble, Macmillan, 1932, p. 44.

Therefore, the rational solutions (ξ, η) of $A\xi^2+\eta^2=1$ are given by

$$\xi=\frac{2\lambda}{\lambda^2+A} \qquad \eta=\frac{\lambda^2-A}{\lambda^2+A} .$$

Setting λ equal to the rational number m/n, where m and n are relatively prime, we may put the solutions in the form

$$\xi=\frac{2mn}{m^2+An^2} \qquad \eta=\frac{m^2-An^2}{m^2+An^2} .$$

When m and A have a greatest common factor α we may reduce the fractions further, byt writing $A=a\alpha$ and $m=m'\alpha$ and obtain the solutions in the form

$$\xi=\frac{2m'n}{\alpha m'^2+an^2} \qquad \eta=\frac{\alpha m'^2-an^2}{\alpha m'^2+an^2} . \tag{1}$$

In case, m', n and A are all odd this will reduce further to the form

$$\xi=\frac{m'n}{\frac{1}{2}(\alpha m'^2+an^2)} \qquad \eta=\frac{\frac{1}{2}(\alpha m'^2-an^2)}{\frac{1}{2}(\alpha m'^2+an^2)} . \tag{2}$$

Moreover, corresponding to the rational solutions (1) of $A\xi^2+\eta^2=1$ we have the primitive integral solutions of $Ax^2+y^2=z^2$

$$x=2m'n \qquad y=\alpha m'^2-an^2 \qquad z=\alpha m'^2+an^2 \tag{1'}$$

and in case m', n, and A are all odd the integral solutions

$$x=m'n \qquad y=\tfrac{1}{2}(\alpha m'^2-an^2) \qquad z=\tfrac{1}{2}(\alpha m'^2+an^2) \tag{2'}$$

where $A=a\alpha$, m', n are relatively prime, and m' and a are relatively prime.

Theoremes de Geometrie Elementaire

Par V. THÉBAULT
Le Mans, France

La présente note a pour but de généraliser une propriété que nous avons signalée il y a quelques années.

Mathesis, (1930), p. 295.

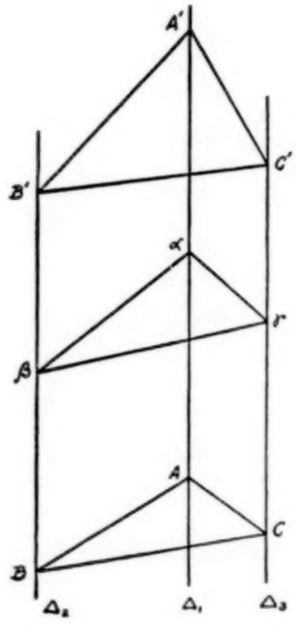


Fig. 1

1. Theoreme 1. *Sur des droites parallèlles Δ_1, Δ_2, Δ_3 d'un même plan, on marque respectivement des points A et A', B et B', C et C' et les points α, β, γ qui divisent les segments AA', BB', CC' dans un même rapport arbitraire k. On a la relation*

$$(k+1)s = S + kS'$$

entre les aires s, S, S' des triangles $\alpha\beta\gamma$, ABC, $A'B'C'$. Pour fixer les idées, nous supposons que les aires S et S' des triangles ABC et $A'B'C'$ sont positives, ce qui exige que les sens de circulation ABC et $A'B'C'$ soient positifs.

Soient α,β,γ les points tels que

(1) $$\frac{\alpha A}{A'\alpha}=\frac{\beta B}{B'\beta}=\frac{\gamma C}{C'\gamma}=k.$$

On a les relations d'aires

(2) $$s=\alpha\beta\gamma=B\beta'A'C'C-(B\beta\gamma C+\beta B'C'\gamma\alpha+B'A'C')$$

Or,

(3) $$\begin{aligned} BB'A'C'C&=BAC+(k+1)\beta B'A'C'\gamma\alpha \\ &=S+(k+1)\beta B'A'C'\gamma\alpha \end{aligned}$$

et

(4) $$B\beta\gamma C=k\cdot\beta B'C'\gamma$$

car on a, par exemple, en vertuede (1),

(5) $$\begin{aligned} k&=\frac{\alpha A+\beta B}{A'\alpha+B'\beta}=\frac{\gamma C+\alpha A}{C'\gamma+A'\alpha}=\frac{B\beta\alpha A}{\beta B'A'\alpha}=\frac{A\alpha\gamma C}{\alpha A'C'\gamma} \\ &=\frac{B\beta\alpha A+A\alpha\gamma C}{\beta B'A'\alpha+\alpha A'C'\gamma}=\frac{B\beta\alpha\gamma CA}{\beta B'A'C'\gamma\alpha}. \end{aligned}$$

Il en résulte donc que

$$\begin{aligned} s&=S+(k+1)(S'+\beta B'C'\gamma\alpha)-(k\cdot\beta B'C'\gamma+\beta B'C'\gamma\alpha+S') \\ &=S+(k+1)(S'+\beta B'C'\gamma\alpha)-[k(\beta B'C'\gamma\alpha+s)+\beta B'C'\gamma\alpha+S'] \\ &=S+k\cdot S'-ks \end{aligned}$$

c'est-à-dire, en définitive, la formule cherchée

(6) $$(k+1)s=S+kS'.$$

Remarques. 1°. Si on marque sur les droites des points α',β',γ' tels que l'on ait

$$\frac{\alpha'A'}{AA'}=\frac{\beta'B'}{BB'}=\frac{\gamma'C'}{CC'}=k$$

ces points sont symétriques de α,β,γ par rapport aux milieux de AA', BB', CC' et si s' désigne l'aire du triangle $\alpha'\beta'\gamma'$

$$(k+1)s'=S'+kS.$$

On a donc, quel que soit k,

$$s+s'=S+S'=\text{constante}$$

2°. Lorsque $k=\frac{1}{2}$, α,β,γ sont les milieux de AA', BB', CC', et

$$s=\tfrac{1}{2}(S+S')$$

Si A, B, C sont collineaires

$$s=k\cdot S'$$

et on retrouve ainsi la propriété que nous avons signalée, (Mathesis, (1939), p. 295). Enfin, quand $k=-S/S'$, $s=0$ et les points α,β,γ sont collineaires.

3°. Dans tout ce qui précède, nous avons supposé les points α,β,γ à distances finies, c'est-à-dire que $k+1\neq 0$, Si $k+1=0$, α,β,γ sont situes a l'infini sur Δ_1, Δ_2, Δ_3.

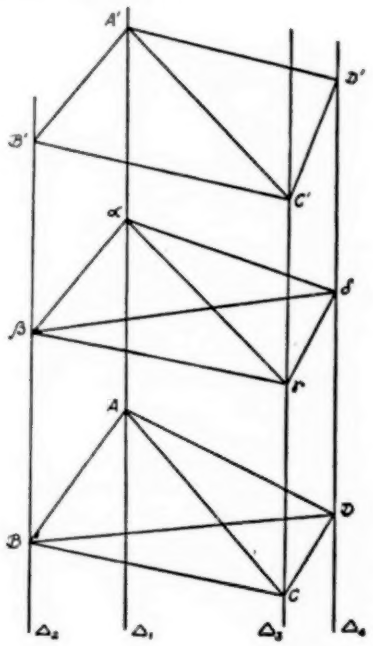


Fig. 2

2. Theoreme 2. *Sur quatre droites parallèles non situées dans un même plan, on marque respectivement des points A et A', B et B', C et C', D et D' et les points $\alpha,\beta,\gamma,\delta$ qui divisent les segments rectilignes AA', BB', CC', DD' dans un même rapport k. On a la relation*

$$(k+1)v=V+kV'$$

entre les volumes v, V, V' des tétraèdres $\alpha,\beta,\gamma,\delta$, $ABCD$, $A'B'C'D'$.

Les points $\alpha,\beta,\gamma,\delta$ sont tels que l'on ait

$$\frac{\alpha A}{A'\alpha}=\frac{\beta B}{B'\beta}=\frac{\gamma C}{C'\gamma}=\frac{\delta D}{D'\delta}=k.$$

Avec les mêmes hypothèses que dans le paragraphe précédent pour les signes des volumes v, V, V' et par un raisonnement identique, il résulte que

$$v=\alpha\beta\gamma\delta=BDCC'B'A'D'-(BDC\gamma\beta\delta+\beta B'D'C'\gamma\delta\alpha+A'B'C'D').$$

Mais, comme $\quad BDCC'B'A'D'=ABCD+(k+1)\beta B'D'C'\gamma\delta\alpha$

et que $\quad BDC\gamma\beta\delta=k\cdot\beta\gamma\delta C'B'D'$

en vertu des relations

$$\frac{BAC\beta\alpha\gamma}{\beta\alpha\gamma B'A'C'}=\frac{A\alpha+B\beta+C\gamma}{A'\alpha+B'\beta+C'\gamma}$$

$$=k=\frac{C\gamma+A\alpha+D\delta}{C'\gamma+A'\alpha+D'\delta}=\frac{CAD\gamma\alpha\delta}{\gamma\alpha\delta C'A'D'}=\cdots,$$

on obtient la formule

$$(k+1)v=V+kV' \tag{7}$$

qu'il s'agissait d'établir.

Remarques. 1°. Certaines valeurs particulières de k donnent dans la formule (7) des résultats curieux.

2°. Les deux théorémes envisagés dans cette note peuvent être énoncés sous une forme plus générale en considérant n droites parallèles, situées ou non dans un même plan, sur lesquelles sont marqués deux groupes de n points A_1 et A_1', $\cdots$, A_n et A'_n.

Edited by
G. WALDO DUNNINGTON

A History of American Mathematical Journals

By BENJAMIN F. FINKEL
Drury College

(Continued from February issue.)*

A comparative view of the various methods by which we arrive at the solutions of questions is at once agreeable and instructive; accordingly, the editor intends to publish as great a variety of good solutions to each question as the limits of the work will admit. This plan will undoubtedly be approved by such as duly consider its numerous advantages.

The editor begs leave to assure the friends of science and of man, that nothing unbecoming a Christian and a gentleman shall be suffered to make its appearance in the work as long as it shall be under his direction. No affected superiority shall be shown, nor *contemptuous* treatment of such as differ from us in opinion, or fall into error. Let a just sense of our own imperfections teach us moderation in our judgment of others; and let us endeavor to show that we are influenced by the noblest motives, the love of elegant and useful science, and the benefit of mankind.

ROBERT ADRAIN.

The last paragraph of the preface, gives the assurance that assumed superiority and the egotistical and discourteous criticisms of the works of others so pronounced under the previous management will no longer be tolerated. (See next to last page Analyst 5).

The whole tone of the *Correspondent* is changed under the new management and its usefulness and reputation correspondingly increased.

The first number of Vol. II of 24 12mo pages, is divided into three articles, as follows: Art. I, "View of the Diophantine Algebra,"...

*The first two paragraphs conclude the preface to the *Mathematical Correspondent*.

continued from Art. XXVI, Vol. I, by Robert Adrain; Art. II, "Observations on the Study of Mathematics", by the editor; and Art. III, "New Questions to be Answered in the Next Number."

The editor, having exhibited in the last number of Vol. I, the principal elementary rules of Diophantine Algebra, exemplifies those rules in his article in this number, by the solution of the five following problems:

I Find two numbers of which their sum and their difference may both be squares.

II To find two such numbers that each added to twice the other may be a rational square.

III To find two numbers such that each added to the square of the other may be a perfect square.

IV To find three numbers such that the square of each added to the sum of the other two may be a perfect square.

V To investigate all possible square numbers such that the sum of any two of them may be a square.

Under I and II, he solves additional problems to illustrate the application of his method of solution.

In his solution of III, he remarks, "The celebrated Euler, in resolving this problem, having arrived at the formula, $u^4-2p^2u^2+u+p^4$, nearly as above, abandons it because, says he, it would be difficult to resolve."

The names of the contributors to number 2 are the following: John Capp, Harrisburg, Pa.; William Charryton, Reading, Pa.; William Child, Pottsgrove, Pa.; John Coope, Chester Co., Pa.; Samuel Cowgill, near Burlington, N. J.; John D. Craig, Baltimore, Md.; James McGinnis, Harrisburg, Pa.; John Gummere, near Burlington, N. J.; Charles Richards, Reading, Pa.; Daniel Smith, Jr., Burlington, N. J.; Seth Smith, Burlington, N. J.; Ebenezer R. White, Danbury, Conn.; John Willits, Mansfield near Burlington, N. J.; in all, a "baker's dozen."

The editor concludes his first article in this number by proposing the following 12 problems which are supposed to be new:

I To find two numbers such that sum of their squares may be a cube and the sum of their cubes a square.

II To find two integers such that the sum of their cubes increased by their product may be a square.

III To find two integers such that the sum of their cubes increased by their product may be a cube.

IV To find two numbers such that their sum is equal to the sum of their biquadrates.

V To find two numbers such that their difference is equal to the difference of their biquadrates.

VI To find the three sides of a rational right angled plane triangle such that the square of each leg increased by the biquadrate of the other two may be a square.

VII To find two numbers such that their sum may be a square, the sum of their squares a square, the sum of their cubes a square, and the sum of their biquadrates a cube.

VIII To find three squares numbers such that each increased by the square of their sum may be a square.

IX To find three or more numbers such, that when each is subtracted from the sum of their squares, the remainders may be squares.

X To find four numbers such that if each be added to, and subtracted from the square of their sum, the sums and remainders may be all squares.

XI To find four integers such that the sum of every two may be a square.

XII To find four or more cubes of which the sum may be a cube.

The editor's second article, "Observations on the Study of Mathematics," is full of practical suggestions some of which are not out of date at the present for the conscientious teacher.

As an example of the multitude of cases in which a knowledge of the fundamental principles of mathematics may be applied with advantage to practical purposes, he says, "It may not be impertinent to give the following: It may be allowed that it is a useful problem in Geography to determine the distance between two places on the surface of the earth."

He then shows how this problem may be readily solved "with great ease and expedition by scale and compass provided that one has a tolerable notion of the first principles of geometry." He proceeds:

"Imagine the two places to lie in the circumference of the base of a hemisphere; and supposing two meridians to pass from these places to one of the poles of the earth, which will be somewhere on the same hemisphere, we shall have given in a spherical triangle the two distances from the places to the pole which are the compliments of their latitudes and the contained angle which is the difference of their longitudes, to find the distance between the places which is the base

of the triangle." By drawing tangents to the meridians at the pole and producing them until they meet the radii of the sphere through the places, we can, from plane trigonometry, find the length of the tangents T and t and the lengths S and s of the produced radii. We then have the four sides of a plane trapezium T, t, S, s and the angle between T and t to construct the trapezium. By constructing the trapezium to any radius, the angle between S and s is found and hence the distance between the two places. Several other equally interesting problems are considered.

The editor's last article in this number "New Questions to be Answered in the next Number" contains nine problems.

The first problem was proposed by William Charryton, Reading; the second by "an Old Soldier"; the third by John Capp, Harrisburg; the fourth by the same; the fifth by Ebenezer R. White, Danbury, Connecticut; the sixth by the same; the seventh by James McGinnis, Harrisburg; the eighth by John Gummere, near Burlington, New Jersey; and the ninth a prize problem by the Editor, reads as follows:

Among the many precious antiquities destroyed by the Caliph Omar in the City of Alexandria was a magnificent temple dedicated to geometry. The edifice consisted of a cylindrical tower one hundred feet in diameter and as many in height with a roof constructed in the following manner: the ridge of the roof was a line directly over a diameter of the upper base of the tower, to which it was both equal and parallel at the distance of fifty feet, and the rafters extending from all points of the ridge to the circular eaves of the upper base were straight lines at right angles to the ridge. As none were permitted to enter this venerable temple of science but such as could determine at least the solid content of the cavity of the roof if not its superfices; the problem is here proposed to the Geometers of America, and he who gives the best investigation of both surface and solidity shall carry off the prize.

Thus concludes this number and with it the *Mathematical Correspondent.*

NOTE: It may be noted that the present generation is witnessing the destruction of such temples of art and culture at the present time by self assumed civilized men who are leaving behind them the ruins of the temples as monuments to their own brutal selfishness and hatred and their unlimited ignorance and stupidity.

Copies of this periodical are very scarce; the only complete file, consisting of Vol. I, and No. 1, Vol. II, to our knowledge, is in the Library of the University of Pennsylvania.

Bolton in his Catalogue of Scientific Periodicals, 1665-1895, gives the following libraries as having Vol. I: Harvard, Yale, and New York Public Library. The copy which was formerly in the Astor Library of New York City is now in the Lenox Library of New York City where were transferred all the scientific Peridodicals of the Public Library in 1895. In 1906, when the material for this History was

being compiled there was a copy of Vol. I in the private library of Artemas Martin, Washington, D. C.

The leading contributors to the *Mathematical Correspondent* were: Robert Adrain; William Lenhart, Baltimore, Md.; John D. Craig, Philadelphia; James North, Philadelphia; Ebenezer R. White, Danbury, Conn.; James McGinnis, Harrisburg, Pa.; Thomas Manghan, Quebec; William Green, New York; J. Temple, New York; W. Folger, Nantucket; W. Thompson, Charleston, S. C.; John Craggs, near Richmond, Va.; John Smithis, Philadelphia; A. Rabbit, Harlim, N. Y.; R. Patterson, Jr., Philadelphia; Rev. T. P. Irving, New Bern, N. C.

THE ANALYST; OR MATHEMATICAL MUSEUM

The second mathematical periodical published in America was, as we have already noted, the *Analyst, or Mathematical Museum*. The following is its title page:

THE ANALYST;
OR
MATHEMATICAL MUSEUM
containing
New Elucidations, Discoveries and Improvements,
In Various Branches of The
Mathematics,
With Collections of Questions
Proposed and Resolved
By Ingenious Correspondents.

VOL. I
Utile Dulci

Philadelphia
Published by William E. Ferrand and Co.
Fay and Kammerer, Printers

1808

The first number of *The Analyst; or Mathematical Museum*; is an exact copy of No. 1, Vol. II of the *Mathematical Correspondent*. The editor makes no statement as to his reason for republishing the number.

The paper and the typographical work of that number was very poor and it is stated in a biographical sketch of Professor Adrain, *Democratic Review*, 1844, Vol. XIV, that this so disappointed him that

he had the entire number republished at his own expense, in Philadelphia, not however as the *Mathematical Correspondent*,—but as the first number of a new periodical, *The Analyst; or Mathematical Museum*.

From the title page, we learn that the *Analyst*; *or Mathematical Museum* was started in 1808 in Philadelphia. It does not seem to be true as stated by Professor Cajori, in his *History of the Teaching of Mathematics in the United States*, that it was the *Analyst* that Professor Adrain started at Reading and republished at his own expense but the first number of Vol. II of the *Mathematical Correspondent*.

After Professor Adrain founded the *Analyst*; *or Mathematical Museum*, he seldom, if ever, in the future, referred to it with its full title, but only as the *Analyst*, of which there were apparently four numbers published.

As we shall see later, the *Analyst* which never had a modifying title, was founded by Professor Adrain, beginning with Vol. I, No. 1, March 1st, 1814.

Further research may disclose the fact that the editor after the first number of the *Analyst*; *or Mathematical Museum*, dropped the latter part of its name, merely calling it the *Analyst*.

In addition to the solutions of the problems proposed in the first number, the second number of the *Analyst;* (*or Mathematical Museum;*) contains the following exponential equations by the editor:

1. Given $x^x = a = 100$, to investigate the value of x.
2. Given $x^{x^x} = a = 123456789$, to find the value of x.
3. Given $(mx)^{(nx)^{rx}} = a$, to investigate the value of x.
4. Given $x^{x^{x^x}} = 10$, to determine the value of x.

He finds the value of x in the first problem to be $x = 3.5972845$ and remarks that the problem is a particular case of the more general problem $(mx)^{nx} = a$. His answer to the second problem is $x = 2.805354$, and to the fourth $x = 1.7343125$. In addition to the solution of the editor's prize problem, the temple dedicated to geometry in Alexandria by John D. Craig of Baltimore, Maryland, two very elaborate solutions are given by the editor himself. At the conclusion of Mr. Craig's solution, the editor remarks, "Thus far Mr. Craig and Mr. Gunmere gave a geometrical investigation of the solidity by the method of ultimate ratios. Messres Capp and McGinnis also determined the solid capacity of the roof by Cavalieri's method of indivisibles; all of these agreeing with Mr. Craig in demonstrating that the solid capacity of the roof is precisely half of the circumscribing cylinder. But notwithstanding the attempts of several contributors, the Editor has

received no true investigation of the surface of the roof, nor even a hint of the very curious and interesting principle on which a true solution must necessarily be founded; although the hope of seeing this important principle developed and exemplified was the only motive which induced him to propose the problem as the subject of the prize." In this remark the editor evidently refers to the well known theorem of Cavalieri.

Number 2 concluded with eleven problems proposed for solution in the next number. The tenth and eleventh were prize problems.

The tenth, proposed by Joseph Clay, Philadelphia, Pa., reads as follows:

Rittenhouse's hygrometer is formed of two thin pieces of wood, of the same uniform breadth, thickness, and length, glued together; the grain of one piece running with the length of the hygrometer, and that of the other with the breadth. The contraction or expansion of the latter piece causes the hygrometer to assume a curvilinear form, and one end of the hygrometer being fixed the degree of moisture is measured on the curve line generated in a plane by the other end. Required the equation, quadrature, and rectification of this curve.

The eleventh problem being the second prize problem for the best satisfactory solution of which the proposer, John Patterson, offers a prize of $10.00 reads as follows:

In order to find the contents of a piece of ground, having a plane level surface, I measured with a circumferentor and chain, the bearings and lengths of the several sides, or boundary lines which I found as follows:

1.	N. 45° E.,	40	Perches,
2.	S. 30° W.,	25	"
3.	S. 5° E.,	36	"
4.	W.	29.6	"
5.	N. 20° E.,	31	" , to the place of beginning.

But upon casting up the difference of latitude and departure, I discovered, what will perhaps always be the case in actual surveys, that errors had been contracted in the dimensions. Now it is required to compute the area of this enclosure, on the most probable supposition of this error.

Number III of the *Analyst* contains 29 pages. It contains solutions of 10 of the problems proposed in Number II. The sixth question proposed by John Eberle, Philadelphia, reads as follows:

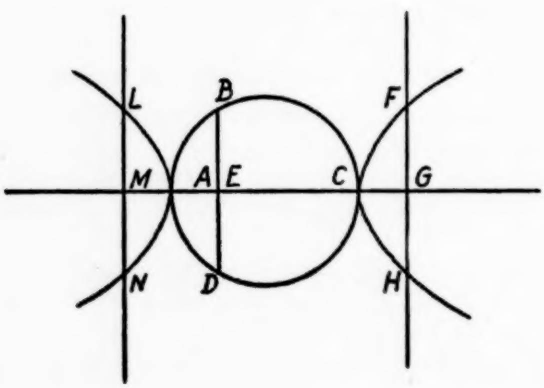
To divide a globe into three equal parts by two parallel planes.

A solution by Charles Richards gives the result, $x = .386963d$, where d is the diameter of the globe and x the height of the segment of one base. Mr. Richards' solution is followed by an interesting Scholium by the Editor.

He says, "I begin with presenting my readers with the following passage from Bossu's *History of the Mathematics*. 'Though few of the works of Diocles have come down to us, we have enough to inform us that he was endowed with great sagacity. Besides his Cissoid, he discovered the solution of a problem which Archimedes had proposed in his treatise on the Sphere and the Cylinder, and which consisted in cutting a sphere by a plane in a given ratio. We know not whether Archimedes himself had resolved this question, considered at that time very difficult, and which leads to an equation of the third order in the modern methods. The solution of Diocles, which is learned and profound, terminates in a geometrical construction by means of two conic sections cutting each other'; to which quotation I add, that the problem may be constructed with greater simplicity by the intersection of one conic section with the circumference of a circle."

He then proceeds to the explanation of the meaning of the three roots of this cubic, which, in this case, are all real. We shall give his explanation and the figure connected therewith.

"On the given straight line AC = unity, as a diameter, describe the circle $ABCD$, and the equilateral or rectangular hyperbola, FCH, LAN; and draw the ordinates FGH, BED, LMN at right angles to MAG; the common property of these curves being that the square of any semi-ordinate, BE^2 or FG^2, is equal to the rectangle of its distance from the vertices A and C, that is, to $AE \cdot EC$ or to $AG \cdot GC$. Now supposing this assemblage of curves to revolve round the straight line MG, let us examine the measures of the generated sphere and hyperbolic conoids.



Put $AE=x$, the solidity of the segment $BAD=S$, that of the globe $ABCD=G$, and that of $FCH=H$; also put p* $=3.1416$. Then $BE^2=x(1-x)$, and dS† $=\pi xdx(1-x)=\pi(xdx-x^2dx)$, which fluxion is evidently positive when $1-x$ is positive, that is, while x is less than unity, and the fluent $S=\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)$ is the true value of the solid having its height $AE=x$. But when x is greater than unity, that is, when AE becomes AG, then the formula, $dS=\pi xdx(1-x)$ is the negative value of the fluxion of the solid FCH, and the correct value of the fluent is $S=\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)=G-H$; so that the formula $\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)$ does not express the sum of the solids $ABCD$ and FCH, but their difference: it is evident, therefore, that the very same expression $\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)$ which is the value of the segment $ABAD$ of the globe, when x is less than the diameter, unity, is also the true value of the excess of the globe above the hyperbolic conoid FCH, when $x=AG$ is greater than the diameter AC. The original problem requiring us to cut off a segment from the globe which should be $=\frac{1}{3}G$ produces the equation $\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$, in which equation there must evidently be one, and only one value of x which answers the question, and is less than AC: but if the question has been, to find AG such that the excess of the globe above the conoid FCG may be equal to $\frac{1}{3}G$, we should have arrived at the very same equation, $\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$, which equation must evidently have one and only one value of x that will answer the question, this value being greater than the diameter AC; we see clearly, therefore, the origin, signification, and use of the two positive roots in the equation, $\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$.

"Again, if we denote AM by x, we have the fluxion of the solid, $LAN=dS=\pi dx(x+x^2)=\pi(xdx+x^2dx)$ and $S=\pi(\frac{1}{2}x^2+\frac{1}{3}x^3)=$ the conoid LAN; and if we suppose this conoid $=\frac{1}{3}G$, we shall have the equation $\pi(\frac{1}{2}x^2+\frac{1}{3}x^3)=\frac{1}{3}G$, in which it is manifest that x must have one, and only one value that will answer the question: and if in this equation we write $-x$ for x we obtain the original equation, $\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$. It is evident therefore that there are three cognate though different problems, each of which leads to the equation, $\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$, and that the three roots of this equation are the answers to the problem, with this condition, that one of these answers coming out negative must have its sign changed.

"There is a certain connection or relation among the answers of these three distinct problems, which deserves notice. Supposing any of the three solids, viz., the conoid LAN, the segment of the globe BAD, or the excess of the globe $ABCD$ above the conoid FCH, to be given

*We shall use π for p.
†Newton's fluxional notation is used by the Editor.

$=\frac{1}{3}G$, and that AM, AE, or AG is required, we shall, in each case, have to resolve the cubic equation $\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$: but if anyone of these three AM, AE, or AG is known, the other two may be found by a quadratic; and if two of the three are known, the third is obtained by a simple equation. This may be completely demonstrated; but it is sufficient at present to be observed, that as these three AM, AE, AG, are the values of x in the equation $\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$, we have only to divide the equation $\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)-\frac{1}{3}G=0$, by $x-r$, r being any known one of the three roots, and the quadratic quotient equated to 0, and resolved will give the other two roots.

"I am obliged to omit several other curious remarks on this subject, and shall content myself with observing that, from what has been said, we discover the value of x that renders $\frac{1}{2}x^2-\frac{1}{3}x^3$ a maximum, in the simplest manner imaginable, x being taken positive, we have only to take such a value that $\pi(\frac{1}{2}x^2-\frac{1}{3}x^3)$ may express the solidity of the whole globe $ABCD$, which is manifestly the case when $x=$ the diameter AC, $=$unity."

We give Professor Adrain's complete discussion to show that nearly one hundred years ago a young American mathematician and teacher was giving far more accurate interpretations of Algebraic results than many of our mathematical teachers of the present day.

The editor states that he received no true solution of problem VII. He then gives his own solution which occupies about $1\frac{1}{2}$ pages. The problem is astronomical and leads also to a cubic equation.

The eighth problem is one in Diophantine Analysis and two solutions, one by Charles Richards, the other by John Capp, are published. The ninth problem is one in surveying and is solved by Daniel Smith.

The prize problem on the Rittenhouse hygrometer is solved by John D. Craig. A solution by Mr. Clay was also furnished the editor. Mr. Clay suggested that the name of the curve be called *Rittenhouse's Hygometrical Curve.*

On page 58, the editor, Dr. Adrain, begins a very interesting discussion on what he calls, "Researches Concerning Isotomous Curves, etc." In this article which covers 10 pages, the *Rittenhouse's Hygrometrical Curve,* comes up as a special *isotomous curve,* in the former the expansion being due to change in moisture and in the latter to change in temperature.

Space forbids further notice at this place of Dr. Adrain's interesting discussion on these curves.

No satisfactory solution having been received of Mr. Patterson's Prize Problem, it was reproposed in this number. However, while the

last pages were going through the press, the editor received a solution from Dr. Nathaniel Bowditch of Salem, Mass.

The contributors to this number were the same as those who contributed to Number 2, with the addition of Jacob Conklin, William Lenheart, Baltimore, Md., and Enoch Lewis, Westown, Pa.

The number concludes with ten problems proposed for solution, the tenth being a prize problem, proposed by John Coope, of Philadelphia and reads as follows:

Given the sides of a trapezium, viz., $AB=6$, $BC=12$, $CD=8$, and $DA=10$; also the sum of the opposite angles ABC and $ADC=300°$; required the area of the trapezium.

The 4th number, the most important of all, because of an article of which we shall speak later, contains the solution of the 10 problems proposed in Number 3.

The prize problem was solved by 16 of the contributors, but only the solution of Nathaniel Bowditch, Daniel Smith, and Joseph Clay were published. The prize of $6.00 was awarded to Joseph Clay. Respecting the prize problem the editor remarks:

"No geometrical construction on this prize question has been received by the editor; he would willingly publish his own method of constructing the problem geometrically, were he not desirous of exciting geometers to a research equally curious and elegant; the question is therefore reproposed in the present number, article 15. After the acknowledgment of solutions received from contributors which now number 29, the editor discusses the 21st and 23rd problems proposed in the *Analyst*. He gives two different solutions of the 21st and generalizes the 23rd. Then follows, pages 88-93, Dr. Bowditch's solution of Mr. Patterson's Prize Problem proposed in the 2nd number, a solution based on the five following assumptions:

1. That the errror ought to be apportioned among all the bearings and distances.
2. That in those lines in which an alteration of the measured distance would tend considerably to correct the error of the survey, a correction ought to be made; but when such alteration would not have that tendency, the length of the line ought to remain unaltered.
3. In the same manner, an alteration ought to be made in the observed bearings, if it would tend considerably to correct the error of the survey, otherwise not.
4. In cases where alterations in the bearings and distance will both tend to correct the error, it will be proper to alter both,

making greater or less alterations according to the greater or less efficiency in correcting the error of the survey.

5. The alterations made in the observed bearing and length of anyone of the boundary lines ought to be such that the combined effect of such alterations may tend wholly to correct the error of the survey.

The editor states, in a subsequent communication, "Mr. Bowditch informed me that he had used this method of correcting surveys several years ago." Dr. Bowditch was awarded for his solution the $10.00 offered by Mr. Patterson.

We now come to the most important contribution published in this journal, and perhaps no article in any subsequent mathematical journals of America had such a direct and immediate bearing on the extension, in a totally new direction, of the boundary of mathematics.

Mathematics is said to be the science of common sense, but it has grown far beyond common sense. It has become a very strange subject from the ordinary point of view, but anyone who penetrates into it will find it a fairyland. From the ordinary point of view, mathematics deals with strange things. There was a Russian peasant who came to Moscow for the first time and went to see the sights. He went to the zoo and saw the giraffes. What did he say? "Look what the Bolsheviks have done to our horses!" That is what modern mathematics has done to simple geometry and simple arithmetic.—From Edward Kasner's *New Names in Mathematics*, one of the Scripta Mathematica Forum Lectures, published by Scripta Mathematica, Yeshiva College, 1937.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

Remedial Reading in College Mathematics

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Introduction. The mortality among college students is high. In the Middle West only about fifty per cent of those who enter college as freshmen remain at the beginning of the junior year. Of course, not all of those who drop out have failed. In recent years the college has become interested in discovering the causes of the many failures. One of the most important findings has been that inability to read is a major cause of failure of college students.(1) Until a few years ago the high school and the college looked upon reading as a subject in which it was the duty of the elementary school to give all the necessary instruction. The realization that even the college must deal with young men and women as they are, and accept the responsibility of being of the greatest possible service to them, finally is emerging. To blame the elementary school or the high school is neither a solution of one of the problems of the college nor a legitimate excuse for failing to try to find a real solution. It is true, however, that the elementary school, where such basic skills as reading are begun, is the most important preparatory agency for the college.

Instruction in Reading. In the primary grades 70 per cent of the time is devoted to reading.(2) In grades IV to VI the emphasis gradually shifts to reading in the content subjects.(3) After grades VI there is little instruction in reading.(4) But all reading is not the same. The reading material of the high school is very different from that of the first six grades. For instance, the reading of algebra involves the pupil in difficulties for which he has not had specific preparation. One of the major difficulties in solving verbal problems is due to reading disability.(5)(6) Reading formulas is difficult by the very nature of

things. Consideration of the long, slow development of mathematical symbolism and the reticence with which men like Descartes adopted the modern exponential notation and other signs should cause a teacher to be sympathetic toward the student who experiences difficulty with the mental processes involved in the comprehension and interpretation of algebraic material. Mathematics is an advanced, condensed and abstract language. If courses in word recognition and interpretation are necessary for college students in English and the social sciences, it is likely that college students of mathematics need instruction in the reading of their subject. Writing of the difficulty of learning to read in the content fields Johnson(7) remarks that students do not and cannot make this transfer without teacher help and guidance.

The Nature of Reading. Reading has been defined(8) as the process of getting meaning from the printed page. It is the art of translating meaningless symbols into meaningful thoughts. For writers the problem is to get as much material as possible on the square inch without sacrificing legibility. Ever since the beginning of the present century interest in reading difficulties has grown. Gray(9) has summarized the investigations of reading and reports over 1800 titles. Only a very small number of the studies have any direct bearing on the special difficulties of reading mathematics. But there is available now a large body of fundamental research in reading that should be understood before attempting instruction in reading in any field. Judd(10) has remarked that college teachers are almost entirely unacquainted with the facts that are known about the reading habits of mature students and that it will be a step in the right direction when college teachers take note of the difficulties inefficient readers encounter.

A great deal of very valuable research of the laboratory type has been devoted to reading. Considerable attention has been given to the study of the eye-movements one makes while reading, and elaborate apparatus has been constructed to photograph eye-movements(2)(4)(11). Much has been learned from these studies. Buswell(4) says that the eyes need training for reading just as the hand does for writing. The records of the eye-movements furnish enlightening evidence with regard to the nature of efficient and inefficient reading. Buswell(4),(12) gives effective graphic and verbal descriptions of good and poor reading. He reports that a first grade girl read 40 words per minute in easy material, with an average of 21 pauses per line, the average pause duration being about 12 twenty-fifths of a second, and with approximately 7 regressive or backward movements of the eyes per line. A college senior read 369 words per minute with an average of

3.6 pauses per line, of average duration 8 twenty-fifths of a second, and without any regressive movements. For first grade pupils the number of fixations per line is about 19 and for college students the average is about 6 per line. It is a long, tedious task to reduce the number of regressions from 2 to 1 per line, and only the most mature readers read with as few as 0.5 regressions per line. It is possible, however, to improve the rhythmic nature of reading even in college.[4] Few people can pronounce more than 250 words per minute. But some can read silently as many as 1000 words per minute. Many grade V pupils read simple material as well as adults. A wide attention span, that is, the ability to hold a large number of words or reading elements in mind at one time, is essential to good silent reading. Gray[18] says that large individual differences in attention span are probably due to training and are not inherent.

Complexity of the mental processes is directly correlated with the complexity of the eye-movements.[14] Adjustment to the difficulty of the reading matter may involve the number of fixations per line, the duration of the fixations, and the number of regressions. There are several kinds of regressive movements. Those that occur near the beginning of a line are usually the result of the faulty backsweep of the eye from the end of the previous line. Within the line they may be due to the difficulty of the words or to the tendency of a good reader to overstep himself occasionally and then be forced to regress in order to get the meaning. In general, confusion in the mind of the reader is reflected in long fixations and many regressions. Reduction in the number of fixations per line is the most common outcome of training the eyes to read. High validity and high reliability in the eye-movement records can be obtained.[15][16][17]

Measuring the rate of reading is one of the simplest ways to determine the maturity of reading ability.[4] The rapid reader is, in general, the one who can reproduce most completely what he reads, that is, the fastest reader usually has the best quality or comprehension in his reading.[11][18][19] Training in comprehension increases the rate of reading. Training to reduce vocalization in silent reading results in a significant decrease in this motor accompaniment and increases the rate of reading.[11] Head movements seem to go with poor reading ability.[11]

Gray[18] lists fourteen reading disabilities and describes exercises to increase the accuracy of recognition and span of recognition. He gives valuable remedial suggestions for dealing with those who have made no progress in reading, those who have difficulty with the mechanics of reading, those who have difficulty in interpretation, slow

silent readers, and those who are weak in all phases of reading. Judd[20] assumes that, with some exceptions, individual differences in reading are due chiefly to acquired methods of attacking the printed page.

Studies of Reading Mathematics. Terry, Rebert and Tinker have made investigations of the eye-movements involved in reading mathematical material. Terry[8] used a group of graduate students as subjects. He made use of their introspections as well as the photographs of their eye-movements. He distinguished two phases of reading, first reading and rereading, and two types of readers, whole first-readers and partial first-readers. Some subjects were predominantly whole first-readers, that is, tried to comprehend every detail of the material in the first reading. Grouping of numerals was found to be an important feature of efficient reading of numerals and seemed to express a natural tendency of the mind to arrange the members of a series of stimuli in groups. The average number of digits included in an eye-pause was about 2.4 and the average duration of the pauses was about 15 twenty-fifths of a second as compared with an average of about 6.5 letters to the fixation and an average pause duration of about 11 twenty-fifths of a second when the same subjects read simple prose. Regressions were more numerous when reading numerals than when reading words. He found that partial and whole first-readers entertain quite different attitudes toward the numerals in a problem that is being read for the first time. Furthermore, they may not be conscious of their peculiar attitude or of the existence of a different attitude. Familiarity and regularity of form were important in determining the number and length of the pauses. For instance, 1000 and 25000 were easier to read than 6783 and 87298 which are of the same number of digits respectively. Some readers used a relatively large number of short pauses in reading numerals and others used relatively small numbers of pauses of relatively long duration. The former seemed to result in the faster reading. He concluded that arithmetic problems and isolated numerals were significantly more difficult as types of reading material than expository prose. When the reading material is difficult the meaning is obtained only after patient and systematic search. Instead of making this effort students often merely glance over the lines or mechanically pronounce the words and get no idea of the thought.

Rebert[21] used three groups of subjects, high school students, chemistry and physics experts, and mathematics experts such as teachers and others who had had several years of graduate study of mathematics. He found that the high school students tended to read the

selections containing algebraic trinomials with more fixations, longer pauses, and more regressions than the experts who were familiar with the content in which the trinomials appeared. However, he concluded that the number of fixations could not be said to be an accurate index in individual cases of ability to read mathematical formulas. There were novices, advanced students and experts who read trinomials with one fixation and there were novices, advanced students and experts who read trinomials with as many as seven fixations. He noted the tendency of all readers to read formulas analytically and in detail rather than as units. Chemistry students read H_2O as they would a word and interpreted it as meaning water but the mathematicians did not read $a^2+2ab+b^2$ as a unit because they did not think of it as representing a mathematical concept. Telling the subjects that they would be asked to reproduce the material made a difference in the eye-movement records. One reader whose mother tongue was not English spent longer durations on the words than on the numerals. Reading mathematical formulas seemed to be like proof-reading where one is continually looking for misspelled words, since even a slight change in a formula may have a very important effect on the meaning. Expressions like πr^2 and 3.1416 were read like words, with only one fixation. Rebert concluded that the presence of formulas in the context tended to cause confusion in the reading record of the immature students, but that regressive movements were not a necessary accompaniment of the reading of formulas. He said that the analytical nature of the reading probably was due to the fact that the formulas were used to express relationships rather than isolated concepts.

Tinker[(2)] made an extensive study of the eye-movements involved in reading formulas. He found that reading a formula which was encountered in algebra made a greater demand on the eyes than did other parts of the same line or other lines which did not contain formulas. He found that there were almost as many fixations in reading a formula as there were characters in the formula. There were regressions in the reading of almost every formula. This he attributed to the analytical method of reading and the slow apprehension rate. The results showed the same tendencies for all subjects regardless of the differences in training. Quadratic trinomials were the most complicated expressions used by Rebert. Tinker used formulas and equations varying from the very simplest to one involving fractions, subscripts and compound literal exponents. He found that fractions and compound exponents made heavy demands on the eyes, as indicated by the number of fixations and the number of regressions. He concluded that training in mathematics makes very little difference in the eye-

movements one uses in reading formulas. There were almost three times as many regressions in the lines of formulas as in the lines devoid of formulas. And there was a 44 per cent increase in the number of fixations when formulas were contained in the lines. The expert mathematicians were more efficient readers of lines of formulas, however, as a group than the high school students. The experts made fewer fixations, shorter pauses and less regressions. Some high school students made pauses of as long as 3 and 4 seconds. Difficulty in apprehending the relations between the parts of the formula was suggested as a probable cause of the long pauses. More formula items were read per pause than words of prose, but 50 per cent more words of prose were read per second than items of formulas. For university students the first and last fixations in a line of prose were centered about 7 millimetres within the line. For algebra containing no formulas they were centered about 4 millimetres within the line. And for a line of formulas they were located practically at the ends of the line, so that in this case the eye covered a great deal more of the line. The length of the eye-movement was only about half as great in reading lines of formulas as it was in reading ordinary prose. Tinker concluded that reading formulas is distinctly different from reading prose, both in the motor phases and in the mental processes involved. This may be the reason that students often tend to pass over formulas they meet in context without reading them. They have acquired the habit of reading straight ahead in prose and attempt to use the same technique in reading material containing formulas. Tinker found very great individual differences among his subjects in reading formulas, and the differences were as large for the naive as for the expert mathematicians.

In a study[22] in which he used the tachistoscopic method of short exposure Tinker found again that individual differences were as pronounced among the expert mathematicians as among the novices. One of the experts was no better at this type of reading of mathematical material than some of the immature students. There were other interesting results in this study too. For formulas within the perception span of the student it was easier for him to read a large number of items in fraction form than in linear form. Fractions seemed to provide a natural grouping schema and configuration became an aid in reading. Also, since the area of vision is a rectangular region, the vertical axis of the eye, though shorter than the horizontal axis, is considerable and increased the number of items that could be read. Signs were easier to read than digits. Digits and letters were of about the same difficulty. The subjects had greater assurance that they were

right in reproducing the material when exponents were not involved. All of the subjects grasped the general form of the formula even if they were unable to reproduce any details. The subjects were able to read more lines exposed tachistoscopically whenever they were able to group them into a regular pattern. He thought configuration accounted for the fact that almost twice as many items could be read in a formula as in unrelated characters.

Tinker[23][24] studied the relative legibility of letters, digits and other signs used in mathematics. Formulas containing compound exponents or subscripts, radicals, parentheses, and the like, hindered easy reading. In tables small figures with white space around them were easier to read than large digits with little space around them. White space was better for separating than lines. Narrow characters like *i*, *l* and *I* were low in legibility. In another study[25] Tinker found that the most important factor affecting the perceptibility of printed characters seemed to be the luminosity difference between the symbol to be read and its background. He found there, too, that there were no significant sex differences in span of visual apprehension.

Comments on the Studies. The teacher will not find many specific recommendations with regard to classroom technique in the teaching of mathematics in the articles of Terry, Rebert and Tinker. It may be that some of these investigators were interested chiefly in developing the eye-movement technique. More significant results might be obtained if a study were made of expert readers of mathematics rather than of expert mathematicians. It must be clear to all that one can be a good mathematician and a poor reader of mathematics. One may compensate for the poor reading by longer and harder study or by high intelligence. Likewise, it would seem that one might be an excellent reader and a poor mathematician. Buckingham[26] reported that the ninth-grade student with the highest algebra score had a reading ability equal to that which is normal for the eighth month of grade IV and an I. Q. of 120. The pupil with the lowest score in algebra in grade IX had a reading ability equivalent to the norm for grade IX and an I. Q. of 98.

A basis for judging the efficiency of reading mathematics might be that the reader should get the meaning from the material as quickly as possible and with as little eye-movement as possible. A study of the eye-movements of expert readers of mathematics and poor readers of mathematics would show the type of maturity of reading toward which all mathematicians should strive. A study of the development of good readers of mathematics from elementary school through college

would be very helpful. In photographing the eye-movements used in reading fractions the vertical as well as the horizontal movements should be recorded.

One wonders, too, whether expert mathematicians have one way of reading for mathematics and another way for simpler material. Or, does the continual reading of mathematics affect all their reading? Judd[20] indicates that analytical reading has a tendency to induce habits of short units of organization. The way people read may not be the way they should read; so just saying this is the way a few expert mathematicians read does not answer the whole question. There is room, then, for further investigation of the type described by Terry, Rebert and Tinker. And there is need for experimentation in the classroom to determine the best ways to teach students to read mathematical material. As investigation continues teachers should try to interpret the results already obtained and apply them until better results are made available. Brain-wave machines are now being introduced to the psychology laboratories and may be even more helpful than the eye-movement cameras have been.

Robinson[27] says that although tachistoscopic tests are not very good measures of perception span they have some value because other measures do not correlate very highly with reading ability.

Difficulties in Reading Mathematics. Frequently one meets college students who are discouraged and confused because they have, as they think, read carefully a page or two of mathematical material and do not understand it. The difficulty may be simply that they failed to realize that a great deal was to be read between the lines and did not supply this matter either in their minds or with pencil and paper. In statistics beginners experience difficulty with the unfamiliar notation and methods of thought representation. Some students have not developed efficient methods for reading theorems with comprehension. And a very common error is the failure to realize the value of reading and grouping into a thought-whole a number of related theorems or other items in a course. Often the reading techniques of students seem to have been picked up heuristically.

Breslich,[5] Stevenson[6] and Buckingham[26] emphasize that comprehension in silent reading is an important factor in success in mathematics. Pressey[28] made a count of the formulas that appear in first year college mathematics, physics and chemistry, and observed that these formulas present reading problems of peculiar difficulty. The article says that often the reason for rote memorizing of formulas is that the student does not know how to read a formula to get its

meaning, and adds that when these formulas appear in the reading material they offer a crucial problem in reading. The article contains a list of the formulas and gives their relative frequencies. The Pressey manual[29] consists of exercises for improving the mechanical habits in reading, for increasing the speed, the comprehension and the ability to grasp long assignments, for finding the main ideas in assignments and for reading tables, graphs and charts.

Judd[30] says that in passing from arithmetic to algebra it is necessary to devise a method of keeping the student's mind free from the habit of attaching particular meanings to the symbols in order that he may concentrate his attention on the mathematical processes of combination. Letters are used for this purpose because nothing has less particular meaning than an isolated letter. But, as all teachers of algebra know, there is danger that the student will proceed immediately to let his mind become so attached to a few letters, such as x, y and z, that he may become one of those who, for instance, can factor x^2-y^2 but cannot factor s^2-t^2, and who think of algebra as "that xyz business." Factoring is a process of finding simple, familiar expressions that are hidden in a problem. Judd[30] says it is like looking for faces in a picture puzzle such as one sees occasionally in the newspapers. The student must be taught to look for simple and familiar expressions. It is useless to expect him to find factors of types that he has never seen.

Techniques for teaching equations and formulas are described by many writers on the teaching of mathematics. Two meanings of an equation should be made clear to the student. An equation may be thought of as a statement to be solved for an unknown. Or it may express a relationship between two or more variables. It is important that the student interpret a formula as the expression of a relationship rather than as made up of abbreviations of the words in a statement. A long list of values might be used to impress the student with the importance of good reading of formulas and equations. Solving verbal problems and the practical problems of life, understanding and appreciating the language of algebra, being able to read intelligently literature involving formulas, using formulas to make thinking more precise and scientific, recognizing the type of equation as an aid in the selection of a method of solving it, understanding the relationship between algebra and geometry by noting that certain forms in algebra correspond to specific loci in geometry, are some of these values. Failure to distinguish between fractions to be combined and an equation involving fractions to be solved is a common error and may be due to inefficient reading. The student should be given experience in enough

situations so that, for instance, he will recognize a quadratic equation even if it involves trigonometric functions, and a hyperbola in the form $pv=k$ as well as $xy=c$ in any context.

One does not need to adopt the extreme view of the Gestalt psychologists with regard to learning by wholes.[31] But there is much in their theory of learning that the teacher of mathematics may use. A whole is more than the sum of its parts. Parts derive their properties from the whole. The whole determines the functions of the parts. Ovbiously no one ever mastered a formula by analysis alone. It is a common occurrence to find college freshmen who see in a formula nothing but a batch of symbols. A student may analyze the formula for the roots of a quadratic equation, but if he stops at that he does not know or understand the formula. For instance, a student may have worked with the quadratic formula for some time and later meet the number $2+\sqrt{3}$ but fail to recognize that it is in the form indicated by the quadratic formula and is a root of a quadratic equation. Thus he loses a valuable part of the understanding of the number system. Tinker[22] found some evidence of the use of configuration in reading formulas and fractions. It seems to the writer that the pattern of a formula should be not only an aid in reading but also of great value in remembering the formula and recalling it and associating it with other things. This can be illustrated for a student by showing him how to associate a formula with descriptive material in which it occurs, by associating a formula with material studied previously, by showing how formulas are essential in the understanding of new principles, by comparing formulas corresponding to two or more tables or graphs and picking out similarities and differences, by analyzing formulas to discover the operations involved in them, and by discussing the meaning and suitability of the names given to formulas.

Reports on Remedial Reading in Colleges. Many colleges are offering remedial reading courses for freshmen and are devising ways of discovering the students who are most in need of instruction in reading. A study[32] at the University of Chicago in 1930 showed wide variations in reading ability among the members of the freshmen class, even in such a carefully selected group. The low rate of reading was made the subject of a remedial program. At Smith College[33] a number of students were found whose rate of reading was about the same as that of a normal reader at the end of eighth grade. Bear[34] reported that at Dartmouth, Chicago, and Smith the slow rate was the outstanding deficiency of the readers. While the fundamental research in reading has made it clear that some of the greatest difficulties

in reading are those encountered in reading mathematics, almost no attention is given to those difficulties by the colleges, if one is to judge by the reports of college remedial reading programs in the literature. The remedial exercises which have been published usually contain a few graphs or charts or tables but one could hardly call that a remedial program for students of mathematics.

Albright and Horning[35] listed the typical reading errors of college students. In descending order of frequency they are: inability to associate related elements of the context, inability to isolate the elements of an involved statement, failure to grasp ideas required in the understanding of later concepts, failure to see the setting of the context as a whole, inability to understand questions, and irrelevant responses. Gerberich[36] reported that the remedial program was helpful to the students on the basis of achievement, probation and remaining in school. Deal[1] reported the results of an experiment which used the control group method in an attempt to develop teaching procedures for the remedial reading work. He used the regular textbooks of the students' courses as reading books and found that some students made a very large gain in comprehension ability. A combination of group and individual instruction was used. Bear[34] described some of the difficulties connected with the inauguration of a remedial reading program. He said that it was already well established that only a small number of practice periods is necessary to produce significant improvement in reading. The Dartmouth plan is somewhat distinctive in that there the interest is in visual defects. Several studies have indicated that visual defects are not important factors in reading. By studying ocular defects and reading disabilities Dartmouth hopes to discover the relationships, if any, between visual deficiencies, reading disability and scholastic achievement. Gray[11] reported that eye defects had no effect on reading ability. The worst case of astigmatism was found in the subject who was the best reader in the group. Dearborn[37] reported that defects in the brain centres account for not more than a small percentage of the reading disability cases.

Jacobson[38] found evidence that reading instruction in the field in which the content was to be mastered was superior to instruction in one course or subject matter field expecting the ability to be transferred to all other fields. Blake and Dearborn[33] cited as one of the values of the remedial work in reading the fact that many of the students became aware for the first time in their lives that it was possible for a deficient reader to improve. Some had thought "once slow, always slow." Blake and Dearborn felt that all remedial work must be pri-

marily individual. Parr[39] found that the students with the highest intelligence profited most from the remedial instruction in reading and that the students with the lowest intelligence profited least. Pressey and Pressey[40] reported that poor reading training could be remedied in a relatively short time and with relatively little drill. In the Chicago study[32] it was found that the reading techniques could be improved by giving definite guidance and that this objective could be attained by using the regular subject matter of a course, with no consequent loss of time to the students. McCallister[41] pointed out that the reading activities necessary for efficient study differ significantly in the various content fields. Each subject employs its own signs, charts and other forms of thought representation and the reading activities are governed by different purposes. Each subject creates needs for special forms of reading ability, and guidance assumes familiarity on the part of the teacher with the peculiar reading demands of the subject.

The Remedial Reading Program in Mathematics. The college remedial reading program should not wait until the student has failed. Each instructor should make a conscious effort early in the term to discover the students who are poor readers of mathematics. Techniques of diagnosis are described by Buswell,[4][12] Gray[18] and others. It should be emphasized that the inauguration of a program of remedial reading for students of mathematics does not require, or even suggest, the installation of eye-movement cameras and other expensive and complicated apparatus. But it is important that teachers be familiar with the general nature of reading, the special reading difficulties of mathematics, and the teaching techniques that will remedy the disabilities. The program should be worked out so that it may become a part of the common practice of the teachers in the daily work of the classroom without interfering in any way with the progress of the study. Buswell[4][12] describes some classroom procedures for remedial work in reading. McCallister[42] gives some suggestions for several content fields, including elementary mathematics. The Thirty-sixth Yearbook of the National Society for the Study of Education, 1937, Part I[43] contains some discussion of the relationship of reading to elementary mathematics and a section on the diagnosis and treatment of extreme reading disabilities. One is not likely, however, to meet a case of dyslexia or congenital word-blindness in a college class, nor even a case of alexia or loss of ability to read because of lesions in the brain caused by disease.

The remedial reading program should be planned as a group activity rather than as an individual job, some allowance being made

for individual guidance. Only extreme cases require individual instruction entirely. Breslich[10] has described a very simple technique that often may be useful in discovering the nature of a student's mathematical difficulties. He asked the student to sit opposite him at a table and to do all his thinking aloud as he worked his problems. Breslich was able in this way to observe the mental processes the student used as well as the errors he made. If one wants a rough description of the eye-movements of a particular student it can be arranged easily to observe the eye-movements either directly or in a mirror. Tinker[16] remarked that since the eye-movement records measure mainly speed of reading it should be more convenient to use standardized reading tests when rate of reading is all that one wishes to determine.

In discussing the reading difficulties of college students Strang[44] points out that inefficient eye-movements are symptoms rather than causes of reading difficulty and that therefore little improvement can be expected by directing the reader's attention to the eye-movements themselves. Judd,[20] also, makes it clear that instruction in interpretation of reading is not the same as instruction in the mechanics of reading. The real processes in reading are those that go on in the mind back of the eye-movements and fixations. The combination of ideas and the reorganization of ideas are the processes with which the teacher is concerned. Judd[10] lists as higher mental processes many which are of paramount importance in the reading of mathematics. Drawing inferences, applying principles, comparing and synthesizing are higher mental processes. But memorizing is not ranked very high.

For improving perceptual span it is not necessary to use an expensive, automatic tachistoscope for accurately timed short exposure of formulas and other reading material. Satisfactory results may be obtained by the use of flash cards and an ordinary watch or a stop watch. Or the material may be flashed on a screen by a projector.

Buswell[12] made a study to determine the reading habits of adults and to learn how to remedy defective reading quickly. In the remedial work the elements emphasized to the students were: (1) the practical value of improved reading habits, (2) the necessity of clear comprehension of what is read, (3) the value of concentration and effort in improving a habit, (4) the importance of getting the thought in large units, (5) the importance of thinking the meaning rather than saying the words, and (6) the value of flexibility, that is, using the best technique of reading under varying conditions. He found that the reading ability of adults is much lower than many teachers of adult classes evidently assume it to be and that the remedial course produced worthwhile results. There are several other very important points in the

report. One is that remedial reading is not simply a matter of giving the students more reading. And another is that the remedial reading program should not overburden itself by emphasizing a large number of details. He says that reading depends on the mastery of a few basic factors and that it is much better to deal with only those factors. Finally, the remedial program should take into consideration the findings of Parr[39] that the most intelligent students benefit most by the guidance in reading.

Summary. Inability to read is an important cause of failure among college students. A remedial reading program seems to be necessary, especially in mathematics. Results of research in the nature of reading and reading disability are presented. Studies of the eye-movements used in reading mathematical material are described. Illustrations of the difficulties encountered in reading mathematics are given. The values reported for college remedial reading programs are reviewed and some recommendations are made in regard to the inauguration of a remedial reading program for students of mathematics. It is suggested that the program become a part of the regular classroom technique of all teachers of mathematics and that an attempt be made early in the term to discover the students most in need of guidance in methods of reading. No elaborate apparatus is necessary and emphasis should be on the few major factors that constitute reading ability. The program should be designed for group instruction, but should be flexible enough to take care of unusual individual cases. It should be available to all students of mathematics,—to the most intelligent as well as others.

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"The vast short-hand symbolism we have invented to record and communicate mathematical ideas is only the *language* of mathematics, and language is a very doubtful asset to any one who has no ideas to communicate." W. B. Carver, in Thinking Versus Manipulation, ***American Mathematical Monthly***, 1937, p. 359.

Mathematical World News

Edited by
L. J. Adams

The Mathematical Association of America will hold its summer meeting at Hanover, New Hampshire on September 9-12, 1940. The Iowa section will meet at Mt. Vernon, Iowa on April 19-20. The Michigan section will meet on April 26-27 at Ann Arbor. The Illinois section will meet at Bloomington on May 3-4.

The summer meeting of the American Mathematical Society will be held at Hanover, New Hampshire in conjunction with the meeting of the M. A. A.

Dr. J. W. Blincoe, University of Tennessee, has been promoted to the rank of assistant professor. Associate Professor J. A. Cooley has been promoted to the rank of professor.

Professor Arnold Emch, University of Illinois, has retired.

Compositío Mathematica, a journal designed to further international cooperation as well as the development of mathematics, is published by P. Noordhoff, Groningen, Netherlands. The price is 20 Dutch guilders per volume. The magazine is under the administration of H. Hopf, G. Julia, and J. M. Whittaker, of Amsterdam, Bruxelles, Zürich, Paris and Liverpool, respectively.

The British Astronomical Association was founded in 1890, and now has over nine hundred members. Its publications include the *Journal*, the *Handbook* and the *Memoirs*, all of which are supplied free to members. Mr. B. M. Peek is president for 1939-40, and Mr. R. M. Fry is editor.

The 1939 volume of Journal de Mathématiques, pures et appliquées, was known as the *Jacques Hadamard Jubilee Volume.* Among the papers it contained was one by Professors T. Y. Thomas and E. W. Titt on *The elementary solution of the general linear differential equation of the second order with analytic coefficients.* The paper was concerned largely with Hadamard's solution of this differential equation as treated in his lectures on Cauchy's problem in linear partial differential equations.

A publication of the U. S. S. R., Moscow, is *Bulletin, Classe des sciences mathématiques et naturelles, série mathématique.*

A recent publication of Osaka Imperial University (Japan) is volume 5 of the collected papers from the faculty of science, *series A· mathematics.*

Volume 48, *Monatshefte für Mathematik und Physik* was dedicated to Ph. Furtwängler in commemoration of his seventieth birthday.

Fascicule XCV and Fascicule XCVI of *Mémorial des Sciences Mathématique* are, respectively: *Theorie de la convergence des procédés d' interpolation et de quadrature mécanique, und Deformation à réseau conjugué persistant et problèmes géométriques qui s' y rattachent.* The former is by M. Ervin Feldheim, and the latter is by M. S. Finikoff.

The January, 1940 number of *The Mathematics Teacher* contained a photograph of Professor David Eugene Smith in commemoration of his eightieth birthday, which occurred on January 21, 1940.

The Southern California Section of the Mathematical Association of America met at Compton Junior Colllege, March 2, 1940. The program was:

1. *Ideal numbers defined over the rational field.* Mr. Walton, San Diego State College.
2. *Junior College Mathematics.* Mr. L. J. Adams, Santa Monica Junior College.
3. *Some æronautical applications of Heaviside's operators.* Dr. W. R. Sears, California Institute of Techonology.
4. *The problem of determining the rank of certain correlation matrices.* Dr. P. G. Hœl, University of California at Los Angeles.
5. *A viewpoint for the curriculum in Secondary Mathematics.* Professor W. L. Hart, University of Minnesota.
6. *Projections in Minkowski spaces.* Dr. F. Bohnenblust, Princeton University.
7. *Generalizations of a formula involving an inverse differential operator.* Professor D. V. Steed, University of Southern California.
8. *The theorem of Jordan-Holder.* Professor Morgan Ward, California Institute of Technology.

Professor G. T. Whyburn, University of Virginia, will present a course in Analytic Topology at the University of California at Los Angeles during the coming summer session.

Dr. A. W. Smith, Colgate University, Hamilton, N. Y. died on February 11, 1940 at the age of 63 years. He taught at Colgate since 1902 and was head of the department since 1920.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

SOLUTIONS

No. 313. Proposed by *C. W. Trigg*, Los Angeles City College.

I. Find a number with three like middle digits whose square is a permutation of the ten digits.

II. Using only two consecutive digits form a five-digit number whose square is a permutation of the ten digits.

Each has a unique solution.

Solution by the *Proposer*.

I. The inequality $31992<N<99381$ is evident. Since N^2 contains each of the ten digits only once, it is a multiple of 9, hence N is a multiple of 3. Then, a and b being consecutive digits, N is a permutation of ***aabbb*** in which a is a multiple of 3, or else N is a permutation of ***abbbb*** where $a+b$ is divisible by 3. Under these restrictions there are 63 eligible values of N. The last four and, in general, the first four digits of N^2 can be read from a table of the squares of numbers less than 10,000. Duplicate digits thus are found in all but the squares of 44445, 76776 and 67677. When these are squared, $(67677)^2=4580176329$ is found to be the unique solution.

II. As above we have $31992<N<99381$ and $N\equiv 0 \bmod 3$. As N^2 may not contain duplicate digits, no digit of N may be zero. Thus the possible values of N are restricted to 186 numbers. A table of squares reveals duplicate digits occurring among the first four and last four digits of the squares of all except ten of the possible values of N. When these are squared, $(97779)^2=9560732841$ is found to be the only solution.

No. 316. Proposed by *H. T. R. Aude*, Colgate University.

If two rectangles $ABCD$ and $EFGH$ are located in the plane, and if the midpoints of the line segments AE, BF, CG, and DH are denoted respectively, I, J, K, L; show that if the figure $IJKL$ is a quadrilateral it is a parallelogram. For two rectangles in fixed positions how many parallelograms may thus be formed?

Solution by *Walter B. Clarke*, San Jose, California.

It is not necessary that $ABCD$ and $EFGH$ be rectangles since the statement is true for parallelograms: Draw AF and DG with M and N as their respective midpoints. Now IM is parallel and equal to half EF; LN is parallel and equal to half GH. But EF is equal and parallel to GH. Thus IM is parallel and equal to LN. Similarly, by referring to AB and CD, we have MJ equal and parallel to NK, whence IJ is equal and parallel to KL. That is, $IJKL$ is a parallelogram.

Concerning part 2, it is possible to interpret the question in more than one way but following the indication, we may draw lines in pairs from A and B to E and F, F and G, G and H, or H and E with corresponding pairs of lines from C and D to G and H, H and E, E and F, or F and G, respectively, which give four parallelograms. Similarly, using B and C with A and D, four more can be obtained. Using C and D with A and B; and D and A with B and C merely repeats the foregoing. The total is thus eight. More can be had by using the diagonals of the parallelograms but this does not seem to be intended from the problem.

No. 317. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

Given three collinear points A, B, C. The perpendiculars BP, CQ are dropped from B and C upon a variable plane passing through A. Find the locus of the point $M=(BQ,CP)$.

Solution by *Paul D. Thomas*, Norman, Oklahoma.

Take any plane through A. Drop the perpendiculars BP, CQ pon the plane and let $M=(BQ, CP)$. Draw 2 lines through M parallel respectively to CQ and AQ. These lines will meet AC in the points T and R respectively. Angle RMT is a right angle by construction. We will prove that M lies on a sphere whose diameter is RT.

Proof: Triangles RMT, APB, AQC are similar.

Hence $$\frac{\overline{RM}^2}{\overline{AP}^2}=\frac{\overline{MT}^2}{\overline{PB}^2}=\frac{\overline{AB}^2}{\overline{AC}^2}, \quad \begin{array}{l}\overline{RM}^2\cdot\overline{AC}^2=\overline{AP}^2\cdot\overline{AB}^2\\ \overline{MT}^2\cdot\overline{AC}^2=\overline{PB}^2\cdot\overline{AB}^2\end{array}$$

From these last two expressions we have

$$\overline{RM}^2+\overline{MT}^2=(\overline{AP}^2+\overline{PB}^2)\cdot\frac{\overline{AB}^2}{\overline{AC}^2}, \quad \overline{AP}^2+\overline{PB}^2=\overline{AB}^2.$$

From the mentioned similar triangles we have

$$\overline{RT}^2=\frac{\overline{AB}^4}{\overline{AC}^2}.$$

Thus $$\overline{RM}^2+\overline{MT}^2=\frac{\overline{AB}^4}{\overline{AC}^2}=\overline{RT}^2.$$

Since $$\frac{\overline{AB}^4}{\overline{AC}^2}$$

is a constant independent of the particular plane chosen we conclude that M lies on a sphere whose diameter is $\overline{RT}$.

NOTE: The points P and Q lie on spheres whose diameters are $\overline{AB}$ and $\overline{AC}$ respectively. The harmonic conjugate of A with respect to the centers S_1 and S_2 of these spheres is the point R. The harmonic conjugate of A with respect to B and C is the point T. The points $U=(RM, BP)$, $V=(RM, CQ)$ also lie on spheres of diameters RB, and RC respectively. The point $W=(TM, AQ)$ lies on a sphere of diameter AT.

Also solved by *Walter B. Clarke*, *C. W. Trigg*, and the *Proposer*.

No. 318. Proposed by *D. L. Mackay*, Evander Childs High School, New York.

If $$\cot A+\cot B+\cot C=\sqrt{3},$$

then triangle ABC is equilateral.

Solution by *B. A. Hausmann*, S. J., University of Detroit, Detroit, Michigan.

Since $$A+B+C=180°,$$

the given relation is

$$\cot (B+C)=\cot B+\cot C-\sqrt{3}.$$

Expressing $\cot(B+C)$ in terms of Cot B and Cot C, we have $\cot B \cdot \cot C - 1 = (\cot B + \cot C)^2 - \sqrt{3}(\cot B + \cot C)$. Expanding and collecting terms, this is

$$\cot^2 B + \cot B(\cot C - \sqrt{3}) + \cot^2 C - \sqrt{3} \cdot \cot C + 1 = 0,$$

or, solved for Cot B:

$$\cot B = \frac{\sqrt{3} - \cot C \pm \sqrt{-(3 \cdot \cot C - 1)^2}}{2} .$$

Since $\cot B$ must be real, $3 \cdot \cot C - 1 = 0$. Hence $C = 60°$. A, B, C play identical roles in the given condition and thus $A = B = C = 60°$.

Also solved by *Walter B. Clarke, D. L. MacKay, Fred Marer*, and *C. W. Trigg.*

No. 319. Proposed by *V. Thébault*, Le Mans, France.

Determine a system of numeration in which a four-digit number of the form *aaab* is the square of a two-digit number *cd*, having given that a and b are consecutive digits, as are also c and d.

Solution by *C. W. Trigg*, Los Angeles City College.

$aaaa = (cd)^2$ is equivalent to

$$ar^3 + ar^2 + ar + b = (cr + d)^2. \tag{1}$$

Clearly $c \geqq a$. First show $b = a+1$. For if $b = a-1$, $d = c \pm 1$, then (1) reduces to $a(r^2+1)(r+1) - 1 = c^2(r+1)^2 \pm 2c(r+1) + 1$, which is impossible since $r+1$ is not a divisor of 2. Next show $d = c-1$. For if $b = a+1$, $d = c+1$, then (1) gives

$$a(r^2+1)(r+1) + 1 = c^2(r+1)^2 + 2c(r+1) + 1,$$

or $a(r^2-1) + 2a = c^2(r+1) + 2c$: thus $r+1$ is a divisor of $2(c-a)$, and since $c < r$, $2(c-a) = r+1$. Hence $2ac - 2a^2 - 2a = c^2 + 1$ or

$$(c-a)^2 + (a+1)^2 + 1 = 0, \text{ which is impossible.}$$

With the above results, (1) becomes $a(r^2-1) + 2a = c^2(r+1) - 2c$. Hence $r+1$ is a divisor of $2(a+c)$, and, since both a and c are less than r, $2(a+c)/(r+1) = 1$, 2 or 3.

With these values for $2(a+c)/(r+1)$, (1) is easily reduced to the following diophantine equations:

$$(c-a)^2 = 2a^2 + (a-1)^2, \tag{2}$$

$$(2c-a)^2 = 3a^2 + 2(a-2)^2, \tag{3}$$

$$(3c-a)^2 = 4a^2 + 3(a-3)^2. \tag{4}$$

Of these, (3) is impossible since a and $a-2$ are not both multiples of 3 and every square is congruent to 0 or 1 mod 3. Solutions for (2) and (4) are obtained by any of the usual methods: the simplest are

$$(54)^2=2223,\ (\overline{39}\ \overline{38}^2)=\overline{33}\ \overline{33}\ \overline{33}\ \overline{34},\ (\overline{65}\ \overline{64})^2=\overline{24}\ \overline{24}\ \overline{24}\ \overline{25},$$

$$(\overline{901}\ \overline{900})^2=\overline{330}\ \overline{330}\ \overline{330}\ \overline{331},$$

corresponding to $r=13$, 47, 177 and 2561, respectively.

Also solved by the *Proposer*, who refers to a more general problem in *Mathesis*, 1938, p. 198.

No. 320. Proposed by *H. T. R. Aude*, Colgate University.

In a triangle ABC the three altitudes are drawn meeting in the point H. The feet of the altitudes are, respectively, D, E, F. On the line segments DH, EH, and FH as diameters circles are drawn. Show that the three common chords of these three pairs of circles are equal. Also show that they are perpendicular, respectively, to the sides of the triangle DEF.

Solution by *Eunice Lewis*, Sapulpa High School, Sapulpa, Oklahoma.

Each side of the orthic triangle EDF will pass through a point of intersection of two of the circles. For, if DE, (a side of the triangle) intersecting two of the circles does not pass through their point of intersection, then there will be two perpendiculars from H to DE, their feet being at the intersection of DE with each circle. This is absurd and the common chords are thus perpendicular to the sides of triangle DEF.

Now the common chords will be equal, for the altitudes of a triangle are the internal bisectors of the angles of the orthic triangle and H is the incenter of DEF.*

Also solved by *Walter B. Clarke*, *B. A. Hausmann*, *D. L. MacKay*, *Paul D. Thomas* and *C. W. Trigg*.

No. 322. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

Find the locus of the point which moves so that its power for a given sphere bears a constant ratio to its distance from a given plane.

Solution by the *Proposer*.

**College Geometry*, Altshiller-Court, Art. 147.

If PN is the perpendicular from the variable point P to the given plane (M), and r is the power of P for the given sphere (A), we have, by assumption, $r = k \cdot PN$.

Consider the coaxal pencil of spheres determined by the sphere (A) and the plane (M). If B, C are two points on the line of centers of this pencil such that $AC = AB = k/2$, the point P lies on one of the two spheres (B), (C) having B, C for centers and belonging to the coaxal pencil considered.* Thus the required locus consists of these two spheres.

If the sphere (A) cuts the plane (M), the two spheres (B), (C) will both be real. Otherwise one of these spheres or both, may be imaginary.

Also solved by *Paul D. Thomas* and *C. W. Trigg.*

No. 325. Proposed by *Walter B. Clarke,* San Jose, California.

Given three points A, B, C. Let D be the reflection of A in BC. Let BD cut the circumcircle of ABC in E. Produce DC to F so that $DC = CF$. Show that FE is perpendicular to BE.

Solution by *C. W. Trigg,* Los Angeles City College.

Draw EC. By symmetry angles BAC and BDC are equal. The theorem may be divided into five cases.

Case I. BD cuts circumcircle again on arc AC.

Case II. BD cuts circumcircle again on arc AB.

Since they are inscribed in the same arc, angles BAC and BEC are equal. Hence angles BDC and BEC are equal, so triangle DCE is isosceles.

Case III. D is outside the circle and BD cuts the circle on arc BC.

Since they are opposite angles of an inscribed quadrilateral, BAC and BEC are supplementary. Also BEC and CED are supplementary. Hence $\angle CED = \angle BAC = \angle BDC$, so triangle DCE is isosceles.

Case IV. D is inside the circle and BD cuts the circle on arc BC.

$\angle BEC$ is the supplement of $\angle BAC$.

$\angle CDE$ is the supplement of $\angle BDC$.

$\angle BEC = \angle CDE$, so triangle DCE is isosceles.

*Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 184, Art. 581. The Macmillan Co., 1935.

In all four cases, $CE=CD=CF$, so E lies on the circle whose diameter is DF, hence DEF is a right angle, so FE is perpendicular to BE.

Case V. D falls on the circumcircle, E coincides with D. Since BC is a diameter, BDC is a right angle, so FE is perpendicular to BE.

Also solved by *Albert Farnell, B. A. Hausmann, Eunice Lewis, D. L. MacKay*, and the *Proposer*.

No. 326. Proposed by *G. W. Wishard*, Norwood, Ohio.

A^2, B^2 and C^2 form an arithmetical progression: find a two-parameter solution for A, B and C in integers.

Solution by *B. A. Hausmann*, University of Detroit.

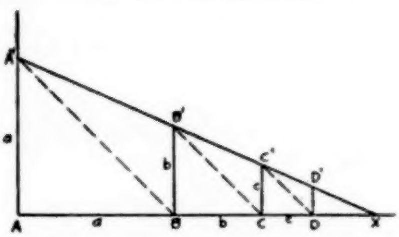
If A^2, B^2, C^2 are in arithmetical progression, $2B^2=A^2+C^2$. Set $A=x+y$, $C=x-y$. Then $B^2=x^2+y^2$. The latter equation has the well-known integral solutions (cf. e. g. Dickson, *Introduction to the Theory of Numbers*, p. 40) $x=2kmn$, $y=k(m^2-n^2)$, $B=k(m^2+n^2)$, where k is arbitrary and m and n are relatively prime. Therefore $A=k(2mn+m^2-n^2)$, $B=k(m^2+n^2)$, and $C=k(2mn-m^2+n^2)$.

Also solved by *Annie Christensen, D. L. MacKay*, and the *Proposer*.

No. 327. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given the line segments a and b, $a>b$, which are the first two terms of a decreasing geometric progression. Give a geometrical construction for the limit of the sum of the infinite progression.

Solution by *C. W. Trigg*, Los Angeles City College.



At the extremities of a line segment $AB=a$ erect perpendiculars $AA'=a$ and $BB'=b$. Extend $A'B'$ to meet AB extended at X. AX is the required sum.

Proof: On AB extended strike off $BC = BB' = b$. Erect a perpendicular to AX at C meeting $A'X$ at C'. Let $CC' = c$. Then $XC : (XC+b) :: c : b$ whence $XC = bc/(b-c)$.

$$(XC+b) : (XC+a+b) :: b : a$$

whence $XC = b^2/(a-b)$. When XC is eliminated, we have $ac = b^2$, so c is the third member of the progression. As $A'XA$ is less than 45°, this process may be continued indefinitely, approaching the point X as a limit.

This construction provides a means of exhibiting any desired number of terms of a geometric progression, as well as for showing the divergence of the geometric series for $r = BB'/AA' \geq 1$.

Also solved by *E. C. Kennedy* and the *Proposer*.

PROPOSALS

No. 345. Proposed by *Daniel Arany*, Budapest, Hungary.

Given two coplanar triangles $A_1A_2A_3$ and $A_4A_5A_6$, and the line p in the plane of the two triangles. Let A_1', A_2', A_3', A_4', A_5', A_6' be the points where the lines A_2A_3, A_3A_1, A_1A_2, A_5A_6, A_6A_4 and A_4A_5, respectively, meet p. Let A_{41}, A_{42}, and A_{43} be the respective intersections of A_1A_4' and $A_1'A_4$; A_2A_4' and $A_2'A_4$; A_3A_4' and $A_3'A_4$. Prove:

(1) That the points A_1, A_2, A_3, A_4, A_{41}, A_{42}, A_{43} lie on a conic.

(2) Two new conics may be formed by substituting the pairs A_5 and A_5'; A_6 and A_6' for A_4 and A_4' in the set of (1). Show that these three conics meet in a point P.

(3) Give a straightedge construction for P.

No. 346. Proposed by *E. C. Kennedy*, Texas College of Arts and Industries.

It is required to determine a function, $f(z)$, of $z = x+iy$ subject to the following conditions. Put

$$\varphi = \lim_{z\to 0} \frac{f(z)-f(0)}{z},$$

where z approaches 0 along the path $x = y^k$. Then for $0<k<1$, $\varphi = 0$; for $k=1$, $\varphi = 1$; $1<k<2$, $\varphi = \infty$; $k=2$, $\varphi = 1$; $k>2$, $\varphi = 0$.

No. 347. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Construct a triangle given a side, the difference between an adjacent angle and the Brocard angle, and the distance of the Brocard point from the given side.

No. 348. Proposed by *Alfred Mœssner*, Nurnberg, Germany.

Find the general solution in integers for the equations

$$A_1+A_2+A_3=B_1+B_2+B_3=X^2,$$

$$A_1\cdot A_2\cdot A_3=B_1\cdot B_2\cdot B_3=Y^2.$$

(For example: $A_1=9$, $A_2=A_3=20$, $B_1=B_2=12$, $B_3=25$, $X=7$, $Y=60$.)

No. 349. Proposed by *V. Thébault*, Le Mans, France.

(1) Given a rectangle $ABCD$. Construct an equilateral triangle AMN whose vertices M and N lie upon BC and CD, respectively.

(2) What is the necessary and sufficient condition that M and N lie between B and C, and C and D, respectively? Under this hypothesis, let P and Q be the intersections of sides AD and AB with the parallels to AB and AD drawn through M and N, and let S be the intersection of MP and NQ. Show that the lines BP, DQ, and CS are concurrent at a point V whose antipedal polygon with respect to the polygon $BCDPSQ$ is a regular hexagon whose area is twice that of triangle AMN and whose circumcircle is tangent at V to the nine-point circle of triangle AMN.

NOTE: The polygon $(Q)=B_1B_2\cdots B_n$, whose vertices are the orthogonal projections of a given point V upon the successive sides of a polygon $(P)=A_1A_2\cdots A_n$, is the *pedal* polygon of V with respect to (P). Inversely, (P) is the *antipedal* polygon of V with respect to (Q).

No. 350. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Show that 5525 is the hypothenuse of twenty-two integral right triangles. Find them.

Bibliography and Reviews

Edited by
H. A. Simmons

Coordinate Geometry. By L. P. Eisenhart. Boston, Ginn and Company, 1939. 11+297 pages. Price $2.50.

This book differs in a number of respects from the conventional American textbook on analytic geometry. The usual aim of providing an adequate foundation for the study of calculus has not been ignored, but a second objective appears to be dominant; namely, to provide a sound foundation for the further study of geometry itself, and to emphasize geometric reasoning in relation to algebraic processes.

A noteworthy feature of this book is the early introduction of solid analytic geometry. A treatment of lines and planes in space is given in the second chapter, following the treatment of points and lines in the plane given in the first chapter. The author makes this procedure especially effective by emphasizing the concept of direction cosines rather than the concept of slope in his discussion of the straight line in the plane. Following the discussion of lines and planes in space, extensions of spaces to four or more dimensions are indicated briefly; and exercises are provided to develop this topic further.

Another feature of this text is the prominent role played by determinants. No previous knowledge of determinants is assumed, however, as they are defined and their properties are derived along with their geometric applications. Determinants of the third and higher orders are defined by means of their expansions in minors.

A third distinctive feature is an appendix to Chapter I, where, by means of Hilbert's fifteen axioms, the correspondence between Euclidean plane geometry and coordinate geometry is indicated. In this connection an interesting discussion of the historical setting of the axioms is given.

The third chapter deals with transformations of coordinates, both in the plane and in space, and contains a brief discussion of polar coordinates. A section which deals with spherical and cylindrical coordinates is included in this chapter. The fourth chapter is devoted chiefly to conics, although there is a brief section which deals with other locus problems of various types. The general equation of the second degree in two variables is treated also in this chapter, together with an analysis of its degenerate cases. The fifth and last chapter deals with surfaces of revolution and quadric surfaces. A detailed analysis of the general equation of the second degree in three variables is given. The invariants associated with this equation are treated and their application to the classification of quadric surfaces is given.

In the preface the author states: "Its practicability as a text has been tested and proved through use during two years in freshman courses in Princeton University". To the present reviewer it seems that the book would be suited best to superior students. It would serve also as an excellent basis for a course in solid analytic geometry in which the concepts of plane analytic geometry could be reviewed and unified from a more advanced viewpoint. Adequate lists of exercises are provided; and they are of the type to develop the student's initiative rather than to provide drill work.

The printing, figures, and other physical characteristics of the book are excellent.

University of Wisconsin. H. P. Evans.

Mathematical Recreations and Essays. (Eleventh Edition). By W. W. Rouse Ball, revised by H. S. M. Coxeter, The Macmillan Co., New York. 1939. xvi+418 pp. Price $2.75.

For nearly half a century this delightful classic has furnished amusement to many who would explore mathematical realms for recreation, and it has stimulated others to attack the unsolved problems of earlier editions and to generalize the results of some of the known theorems. It is thus a real event to have at hand a new edition published 17 years after the tenth, which appeared in 1922.

The revison has been extensive, although there has been no essential change in the character of the material included in the volume. The fifth chapter of the old edition on mechanical recreations, the eighth on bees and their calls, and the fifteenth on string figures have been omitted. The twelfth, devoted to miscellaneous problems (Chinese rings, problems connected with packs of cards, etc.), has been broken up and distributed among other chapters. The fourteenth, on cryptographs and ciphers, has been completely rewritten by Abraham Sinkov, a cryptanalyst in the U. S. War Depatrment. The new material is found in arithmetical recreations, Chapter 2; geometrical recreations, Chapter 3; polyhedra, Chapter 5, which is entirely new; magic squares, Chapter 7; and map-coloring problems, Chapter 8. Chapter 5 is handsomely illustrated by two half-tone plates and numerous figures, which greatly illuminate the text.

The reviewer does not regret the omissions which the revisor has found necessary to make, since these recreations are still available in the tenth edition, reprinted last in 1937, and the new material adds greatly to our ever growing fund of problems which belong to the realm of mathematical esthetics. One has but to turn to earlier editions than the tenth to note this increasing supply of recreations. The reviewer still recalls the interest which he found in a chapter on the nature of the ether (if there is an ether), which has long since disappeared from new editions.

It would be impossible to attempt to describe the new material in detail, or even to mention the many new topics that have been introduced. One notes with interest a considerable increase in material derived from the theory of numbers, and the numerous problems in which techniques derived from the theory of continued fractions are employed. One notes with interest a competent account of the problem of the distribution of prime numbers, a good description of Lehmer's ingenious machine for factoring large numbers, a device which the author characterizes as "this wonderful machine", and the theory and application of Fibonacci numbers. One also finds among the new sections, accounts of the minimal problems of Besicovitch and Kakeya, the Platonic and Archimedean solids, zonohedra, the theory of the kaleidoscope, the present status of the map-coloring problem, the problem of coloring the icosahedron, tesselation, or the covering of a plane area with mosaics, problems in probability, continued fractions and lattice points, magic domino squares, magic cubes, etc.

It is also interesting to note the progress made with certain problems. One of the most interesting of these is that connected with Mersenne's mysterious conjecture that numbers of the form 2^p-1 are prime for values of p less than or equal to 257 only when p is one of the numbers 1, 2, 3, 5 7, 13, 17, 19, 31, 67, 127, and 257. In earlier editions an entire chapter was devoted to this puzzle, but in the tenth edition it was given a scant two paragraphs since certain errors tended to discredit Mersenne's statement. In the new edition, this old problem gains new space since recent investigations have once more supported Mersenne's conjecture.

If one were to make a criticism of the new edition, it would be that the new material is in many instances considerably more difficult than that which it has displaced, and to this extent the book is less useful to beginning mathematical students. For example,

the account which the author gives of the minimal problem of Kakeya is almost unintelligible without reference to the original sources. Because of the difficulty of the solution of the problem, only the vaguest account can be given of the argument. As science, and particularly mathematical science advances, the problems and discoveries tend to grow in complexity. Hence, the simpler questions of an earlier year, still novel to beginning students, no longer have the same interest for modern mathematicians. But since Birkhoff's formula for beauty contains "complexity" in its denominator, one cannot but feel that the newer recreations have lost a little of the charm of earlier ones.

Northwestern University. H. T. DAVIS.

Descriptive Geometry. By A. V. Millar and K. G. Shields. D. C. Heath and Company, 1939. x+192 pages, $2.25.

A few lines should suffice to characterize for the readers of a mathematical periodical a book as purely ultilitarian in its spirit as is this little book. Its aim is the preparation of engineering draftsmen. In procedure its authors exhibit particular regard for commercial practice, as shown by the exclusive use of the third quadrant, the complete elimination of the ground line from all figures, and emphasis on developments and other applications, e. g., in geology and mining. There is equal disregard for standard geometrical language and mathematical precision, as illustrated in the use of the word *line* for *curve*, the reference to points and lines as *magnitudes*, the use of the verb *touch* when *intersect* is meant, the reference to the prism of three faces (including the bases), and the definition of a plane figure as a *geometrical configuration which lies wholly in a plane.* The fourteen-page chapter on shades, shadows, and perspective emphasizes the same utilitarian aspects and, in the reviewer's opinion, is too sketchy to be of much value.

This criticism should not be taken as implying that the book is unsuited to its purpose; the authors claim to have used it with marked success and the drawing departments of other engineering colleges may well have similar experience. The figures are clear, the typography excellent, the general format pleasing. But it will contribute nothing toward raising descriptive geometry in this country to the place of high esteem which it holds in certain European countries.

University of Michigan. JOHN W. BRADSHAW.

College Algebra. By H. L. Rietz and A. R. Crathorne. Henry Holt and Company, New York. Fourth Edition, 1939. xv+298 pages. $1.85.

This is the fourth edition of the well-known textbook, a new edition of which has appeared every ten years since 1909. It is by far the most extensive revision made; a fact that is made evident, even before opening the book, by the larger page-size ($5\frac{1}{4}$ by $8\frac{1}{4}$). The type is larger and clearer, and the margins are wider. Bold face, or black, type is used more consistently for emphasizing words, phrases, and formulas. A new feature is the addition of a number of Oral Exercises, which are numbered separately from the Written Exercises. This separate numbering results in at least one possible confusion, for, on page 27 a reference is made to "Prob. 7, Art. 15". Article 15 is followed by eight "Oral Exercises" and seven "Written Exercises"! The number of exercises is considerably increased. The articles in a number of chapters have been rearranged for better presentation.

Considerable additional material has been added to the subjects of factoring, fractions, and the theory of equations. Exponents and radicals have been collected into a new chapter. New articles have been added on ratio and proportion, solution of three equations in three unknowns by elimination, and on approximate numbers and computation. The section on resultants is omitted.

Criticisms, mostly minor in character, would include the following. The rectangular coordinate system is introduced, and the negative $X-$ and $Y-$axes are labeled X' and Y', respectively Later figures are not consistent with this, as some have the X' and Y' marked, and others have not. The graphs in figures 15 and 19 are poorly drawn. On pages 80-81, in showing that *A quadratic equation has only two roots*, no mention is made of the possibility of a third root being the same as one of the two roots given by the quadratic formula. The same defect occurs in the chapter on the theory of equations in discussing the number of roots of the equation of nth degree. On page 132, the statement "In a number system thus extended [rational number system] an equation $ax+b=0$, where a and b are any integers or fractions, has a solution" is made. The restriction $a\neq 0$ should have been made. On page 185, after careful use of inequalities, there is the statement "log $y=n-1+$(a positive fraction)". The word "non-negative" should replace "positive".

Misprints seem to be very few, and numerous misprints in the third edition have been corrected. The book has been carefully revised, and, in a great many respects, may be considered as a new book. The binding, both as to appearance and construction, has been greatly improved. All the topics usually considered as parts of college algebra are included.

University of Arkansas. EDWIN COMFORT.

LITERATURE RECEIVED BY THE EDITORIAL BOARD DURING PERIOD*

JULY 1, 1938 - JULY 1, 1939
(Volume XIII)

(1) *Advanced Mathematics for Engineers.* By H. W. Reddick and F. H. Miller. John Wiley & Sons, Inc., New York, 1938. x+473 pages. $4.00.

(2) *Advanced Analytic Geometry.* By Alan D. Campbell. John Wiley & Sons, Inc., New York, 1938. x+310 pages. $4.00.

(3) *Advice to the Graduate Assistant.* By A. D. Campbell. Reprint from the *American Mathematical Monthly*. Vol. XLV, No. 1, January, 1938.

(4) *Analytic Geometry.* By Roscoe Woods. The Macmillan Co., New York, 1939. xiii+294 pages. $2.25.

(5) *Analytic Geometry and Calculus.* New Edition. By Woods and Bailey. Ginn and Co., New York, 1938. xi+487 pages.

(6) *An Introduction to Modern Geometry.* By Levi S. Shively. John Wiley & Sons, Inc., New York, 1939. xi+167 pages. $2.00.

*The April issue will contain lists covering the period to the present date from July 1, 1930.

(7) *Applied Mathematics in Chemical Engineering.* By Thomas K. Sherwood and Charles E. Reed. McGraw-Hill Book Company, Inc., New York, 1939. xi+403 pages.

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(48) *Isaac Newton, 1642-1727.* By J. W. N. Sullivan. The Macmillan Co., New York, 1938. xx+275 pages. $2.50.

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